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CIE A-LEVEL FURTHER MATHS 9231 (FM)

FORMULAE & SOLVED QUESTIONS FOR FURTHER MECHANICS

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1. Calculus in Mechanics

1.1 Acceleration is a Derivative

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = v.\frac{dv}{ds}$$

Example:

A particle moves with acceleration, $a = -2v^2$ where v is velocity. Initially, particle at 0 with $v = 2$

i. Find expression for v, in terms of s

ii. Find expression for v, in terms of t

Solution:

Part (i):

Express acceleration as a derivative with displacement

$$v.\frac{dv}{ds} = -2v^2$$
$$v.dv = -2v^2.ds$$

Separate the variables and integrate

$$\int \frac{v}{v^2}.dv = \int -2.ds$$
$$\ln v = -2s + c$$

Substitute given information to find c

$$\ln 2 = 0 + c \therefore c = \ln 2$$

Rearrange found equation to make v the subject:

$$\ln v = -2s + \ln 2$$
$$\ln v - \ln 2 = -2s$$
$$\ln\left(\frac{v}{2}\right) = -2s$$
$$\frac{v}{2} = e^{-2s} \therefore v = 2e^{-2s}$$

Part (ii):

Express acceleration as a derivative with time

$$\frac{dv}{dt} = -2v^2$$
$$v^{-2}.dv = -2.dt$$
$$\int v^{-2}.dv = \int -2.dt$$

Integrate the expression

$$-v^{-1} = -2t + c$$

Substitute given information to find c

$$-(2)^{-1} = -2(0) + c \therefore c = -\frac{1}{2}$$

Rearrange found equation to make v the subject:

$$-v^{-1} = -2t - \frac{1}{2}$$
$$v^{-1} = 2t + \frac{1}{2}$$
$$v = \frac{1}{2t + \frac{1}{2}} = \frac{2}{4t + 1}$$

1.2 Variable Forces

- Bodies undergo variable acceleration due to the effect of variable forces e.g. gravitational fields
- Important to put negative sign if decelerating e.g. when a body falls in resistive medium
- Deriving an expression for force in terms of velocity and displacement:

$$F = ma \qquad a = v\frac{dv}{dx}$$
$$F = mv\frac{dv}{dx}$$

- Deriving an expression for force in terms of work done:

For an object moving from x_1 to x_2 the change in kinetic energy or work done can be defined as:

$$W = \int_{x_1}^{x_2} F\,dx$$
$$\therefore \frac{dW}{dx} = F$$

- Deriving an expression of power in terms of force and velocity

$$P = \frac{dW}{dt} = \frac{dW}{dx} \times \frac{dx}{dt} = Fv$$

(IM) Ex: 9Misc **Question 10:**

A car of mass 1200kg is travelling on a straight horizontal road, with its engine working at a constant rate of 25kW. Given that the resistance to motion of the car is proportional to the square of its velocity and that the greatest constant speed the car can maintain is 50ms^{-1}, show that $125000 - v^3 = 6000v^2\frac{dv}{dx}$, where v ms^{-1} is the velocity of the car when its displacement from a fixed point on the road is x metres.

Solution:

Write down known facts:

$$P = 25000 \qquad F = kv^2 \qquad m = 1200$$

Find k:

$$F = kv^2$$
$$\frac{P}{v} = kv^2$$
$$\frac{25000}{50} = k(50^2)$$
$$k = 0.2$$

Write an equation that relates the forces:

$$Engine\ force = Work\ force + Resistive\ force$$
$$\frac{P}{v} = mv\frac{dv}{dx} + kv^2$$
$$\frac{25000}{v} = 1200v\frac{dv}{dx} + \frac{1}{5}v^2$$

$$\frac{125000}{v} = 6000v\frac{dv}{dx} + v^2$$
$$125000 = 6000v^2\frac{dv}{dx} + v^3$$

Hence find distance covered by the car in increasing its speed from 30 ms^{-1} to 45 ms^{-1} by forming an integral to define displacement in terms of velocity:

$$\frac{dv}{dx} = \frac{125000 - v^3}{6000v^2}$$
$$\frac{dx}{dv} = \frac{6000v^2}{125000 - v^3}$$
$$dx = \frac{6000v^2}{125000 - v^3}dv$$
$$x = \int \frac{6000v^2}{125000 - v^3}dv$$

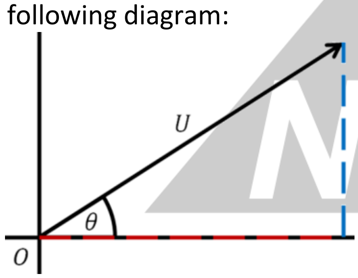
Substitute the info we know:

$$x = \int_{30}^{45} \frac{6000v^2}{125000 - v^3}dv$$
$$x = 6000\int_{30}^{45} \frac{v^2}{125000 - v^3}dv$$
$$x = 6000\left[-\frac{\ln(125000 - v^3)}{3}\right]_{30}^{45}$$
$$x = 2125$$

2. Projectiles

2.1 Initial Velocity

- Consider the following diagram:



- This shows that an initial speed U can be defined in vector notation:

$$\mathbf{u} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j}$$

- In this notation:
  - θ defines the angle of elevation from the horizontal
  - $\mathbf{i}$ defines the horizontal unit vector
  - $\mathbf{j}$ defines the vertical unit vector

2.2 Acceleration

- All projectiles are affected by gravity
- Gravity effects only the vertical component of speed
- We can thus define acceleration in vector notation:

$$\mathbf{a} = 0\mathbf{i} - g\mathbf{j} = -g\mathbf{j}$$

- In this notation:
  - g is the acceleration due to gravity
  - $\mathbf{i}$ defines the horizontal unit vector
  - $\mathbf{j}$ defines the vertical unit vector
- Note: this model assumes that air resistance is negligible

2.3 Final Velocity

- This defines the velocity of the projectile after t seconds of travel
- Final velocity, according to the SUVAT equations, is also dependent on the initial velocity and any acceleration acting on the projectile:

$$v = u + at$$

- Above is a scalar equation which can be written as a vector equation:

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

- This can be further simplified into vector notation by substituting previous equations:

$$\mathbf{v} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j} - gt\mathbf{j}$$
$$\mathbf{v} = U\cos\theta\,\mathbf{i} + (U\sin\theta - gt)\mathbf{j}$$

- Substituting any value of t into the above equation produces the velocity of the projectile at that time

2.4 Displacement

- This defines the distance the particle has moved from the origin
- Displacement, according to the SUVAT equations, is dependent on initial velocity and acceleration, or final velocity and acceleration:

$$s = ut + \frac{1}{2}at^2$$
$$s = vt - \frac{1}{2}at^2$$

- These are scalar equations and can also be written as vector equations:

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{r} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

- Substituting previous equations into either of the above equations defines displacement in vector notation:

$$\mathbf{r} = (U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j})t - \frac{1}{2}gt^2\mathbf{j}$$
$$\mathbf{r} = (Ut\cos\theta)\mathbf{i} + \left(Ut\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}$$

- Substituting any value of t into the above equation produces the displacement of the projectile at that time

2.5 Special Events

- **Time at Maximum Vertical Height**: this is the time at which the projectile reaches its max height, at a given initial velocity and angle of elevation
  - This occurs when vertical component of velocity is 0:

$$U\sin\theta - gt = 0$$

$$\therefore \mathbf{TaMVH} = \frac{U\sin\theta}{g}$$

- **Maximum Vertical Height:** this is the maximum height the projectile reaches during its flight
  - This occurs when vertical component of velocity is 0:

$$\mathbf{MVH} = U(\mathbf{TaMVH})\sin\theta - \frac{1}{2}g(\mathbf{TaMVH})^2$$

- **Horizontal Range**: this is the horizontal distance that the projectile covers at a given initial velocity and angle of elevation
  - This occurs when the vertical displacement is 0:

$$Ut\sin\theta - \frac{1}{2}gt^2 = 0$$

  - Find t and substitute into the equation below:

$$\mathbf{HR} = Ut\cos\theta$$

- **Maximum Horizontal Range**: this is the MAXIMUM possible horizontal distance that the projectile can cover at a given initial velocity
  - This occurs when the angle of elevation is 45°:

$$\mathbf{MHR} = Ut\times\frac{\sqrt{2}}{2}$$

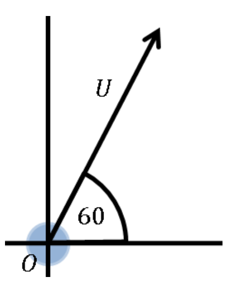
(IM) Ex 7C: **Question 9:**

A ball was projected at an angle of 60° to the horizontal. One second later another ball was projected from the same point at an angle of 30° to the horizontal. One second after the second ball was released, the two balls collided. Show that the velocities of the balls were 12.99ms^{-1} and 15ms^{-1}. Take the value of g to be 10ms^{-2}.

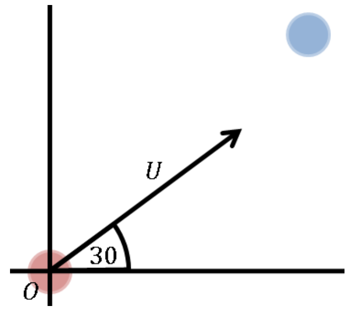
Solution:

Visualise the scenario:

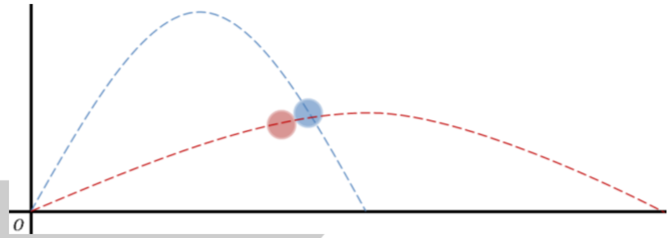
At zero time:



At $t = 1$:



At $t = 2$:



Let velocity for first ball be U_1 and second ball be U_2

Displacement from origin is equal for both at impact

$$\therefore \mathbf{r}_1 = \mathbf{r}_2$$

$$\mathbf{r}_1 = (U_1(2)\cos 60)\mathbf{i} + (U_1(2)\sin 60 - 5(2)^2)\mathbf{j}$$

$$\mathbf{r}_2 = (U_2(1)\cos 30)\mathbf{i} + (U_2(1)\sin 30 - 5(1)^2)\mathbf{j}$$

Equate horizontal components:

$$\therefore 2U_1\cos 60 = U_2\cos 30$$

$$U_1 = \frac{\sqrt{3}}{2}U_2$$

Equate vertical components and substitute info above:

$$\therefore 2U_1\sin 60 - 20 = U_2\sin 30 - 5$$

$$\left(2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\right)U_2 = 15$$

$$U_2 = 15\text{ms}^{-1}$$

Find U_1

$$U_1 = \frac{\sqrt{3}}{2}(15) = 12.99\text{ms}^{-1}$$

3. Turning Effects of Forces

3.1 Moment of a Force

$$\text{Moment of a force} = |\mathbf{F}|\times d$$

$|\mathbf{F}|$: magnitude of the force

d: perpendicular distance from pivot to point of force

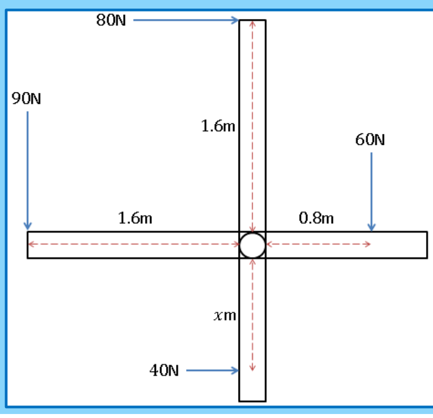
- Units: Newtonmetre (Nm)
- Moments are vector quantities and act clockwise or anticlockwise around pivot
- Clockwise is generally considered as positive direction

- **Principle of Moments:** when a system is in equilibrium the sum of anticlockwise moments is equal to the sum of clockwise moments

anti-clockwise moments = clockwise moments

(IM) Ex 10A: **Question 4a:**

The diagram shows an aerial view of a revolving door. Four people are exerting forces of 40N, 60N, 80N and 90N as shown. Find the distance x if the total moment of the forces about O is 12Nm



Solution:

Use the sum of turning forces equation:

clockwise moments + anticlockwise moments = 12Nm

Find clockwise moments:

$$(80\times1.6)+(60\times0.8)=128+48=176$$

Find anticlockwise moments:

$$-\big((90\times1.6)+(40\times x)\big)=-(144+40x)$$

Substitute back into formula and solve for x:

$$176+\big(-(144+40x)\big)=12$$
$$176-144-40x=12$$
$$x=0.5\text{m}$$

3.2 Unknown Forces

- Magnitude of forces may not always be given
- Eliminate an unknown/unwanted force by making the point on which it acts the pivot

(IM) Ex 10A: **Question 11:**

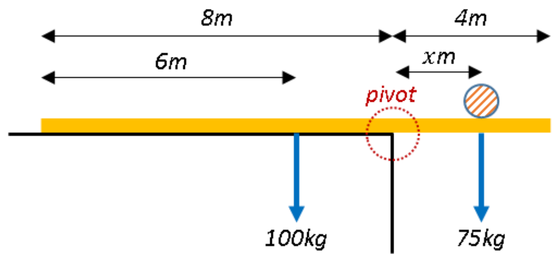
A uniform plank is 12m long and has mass 100kg. It is placed on horizontal ground at the edge of a cliff, with 4m of the plank projecting over the edge.

i. How far out from the cliff can a man of mass 75kg safely walk?
ii. The man wishes to walk to the end of the plank. What is the minimum mass he should place on the other end of the plank to do this?

Solution :

Part (i)

Draw up diagram of given scenario:



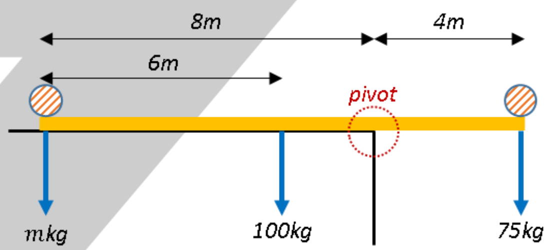
Use the principle of moments and solve for x:

anti-clockwise moments = clockwise moments

$$100g\times2=75g\times x$$
$$\therefore x=\frac{100g\times2}{75g}=2.67$$

Part (ii)

Draw up diagram of given scenario:



Minimum mass therefore maximum distance from pivot

Use the principle of moments and solve for x:

anti-clockwise moments = clockwise moments

$$(100g\times2)+(mg\times8)=75g\times4$$
$$\therefore m=\frac{(75g\times4)-(100g\times2)}{g\times8}=12.5$$

3.3 Forces in Different Directions

- Forces may act at an angle to the plane
- Equilibrium maintained using components of forces

(IM) Ex 10A: **Question 9:**

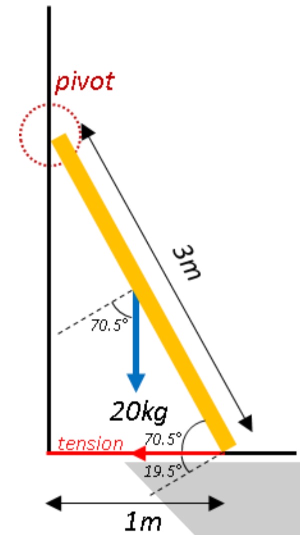
A uniform ladder of mass 20kg and length 3m rests against a smooth wall with the bottom of the ladder on smooth horizontal ground and attached by means of a light inextensible string, 1m long, to the base of the wall

i. Find the tension in the string.
ii. If the breaking strain of the string is 250N, find how far up the ladder a man of mass 80kg can safely ascend.

Solution:

Part (i)

Draw up diagram of given scenario:



Find the angle of elevation, θ, from the horizontal:

$$\cos\theta = \frac{1}{3}$$
$$\theta = 70.5°$$

Use the principle of moments and solve for T:

$$1.5 \times 20g \times \cos 70.5 = 3 \times T \times \cos 19.5$$
$$T = 34.7$$

Part (ii)

In above scenario assume that the man can take any position on the ladder, call it x

Use the principle of moments and solve for x:

$$\text{anti-clockwise moments} = \text{clockwise moments}$$
$$\big((1.5)20g + (x)80g\big)\cos 70.5 = (3)(250)\cos 19.5$$
$$x = 2.32$$

4. Momentum & Impulse

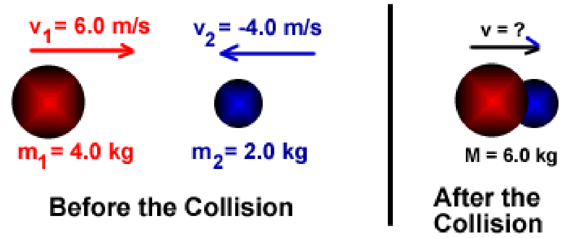
4.1 Momentum

$$\text{Momentum} = mv$$

- **Conservation of linear momentum:** The total momentum of a system in a particular direction remains constant unless an external force is applied

Momentum Before Collision = Momentum After Collision

For example: find the speed after the collision



$$\text{Momentum before} = (6\times4) + (-4\times2) = 16$$
$$\text{Momentum after} = 6\times v = 6v$$
$$16 = 6v$$
$$v = 2.67 \text{ ms}^{-1}$$

4.2 Impulse

$$\text{Impulse} = \text{change of momentum} = mv - mu$$
$$\text{Impulse} = \text{force} \times \text{time}$$

Impulse of a variable force:

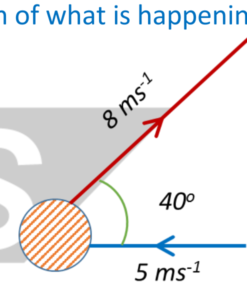
$$\text{Impulse} = \int_0^t F\,dt$$

(IM) Ex 13A: **Example 4:**

Steve kicks a ball mass 0.8kg along the ground at a velocity of 5ms^{-1} towards Monica. She kicks it back towards him, lofting it so that it leaves her foot at 8ms^{-1} and with an elevation of 40° to the horizontal. Find the magnitude and direction of the impulse from her kick.

Solution:

Sketch a diagram of what is happening



Find the initial momentum

$$x\text{-direction} = 0.8\times -5 = -4$$

Resolving the final velocity

$$x\text{-direction} = 0.8\times8\times\cos 40 = 4.903$$
$$y\text{-direction} = 0.8\times8\times\sin 40 = 4.114$$

Find the change in momentum i.e. impulse

$$\text{Impulse} = (4.903\mathbf{i} + 4.114\mathbf{j}) - (-4\mathbf{i})$$
$$= 8.903\mathbf{i} + 4.114\mathbf{j}$$

Hence find the magnitude and direction

Magnitude: $\sqrt{8.903^2 + 4.114^2} = 9.808$

Direction: $\tan\theta = \frac{4.114}{8.903}$ hence $\theta = 24.8°$

4.3 Newton's Law of Restitution

When two objects collide directly,

$$\frac{Seperation\ Speed}{Approach\ Speed} = e$$

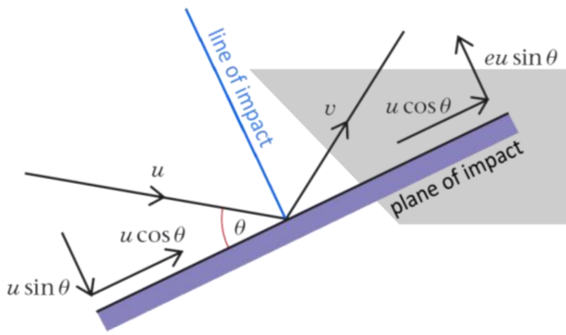
- The constant e is called the **coefficient of restitution** for the objects and takes a value between 0 and 1
  - $e = 0$; totally inelastic impact, no rebounding
  - $e = 1$; perfectly elastic impact, speeds unchanged

In reality, perfect elasticity does not occur hence

$$0 \leq e < 1$$

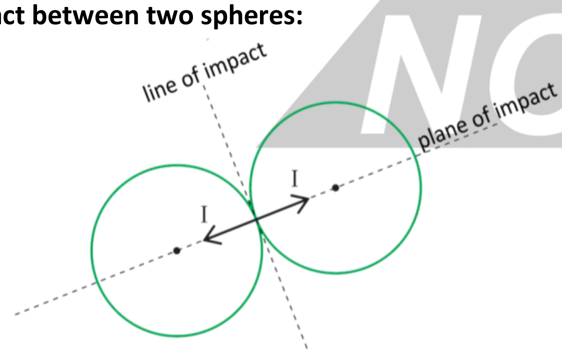
4.4 Oblique Impacts

Impact between a smooth sphere and a fixed surface:



- The impulse on the sphere acts perpendicular to the surface, along line of impact
- Newton's law of restitution applies to the component of the velocity of the sphere along line of impact
- The component of the velocity of the sphere along plane of impact is unchanged.

Impact between two spheres:



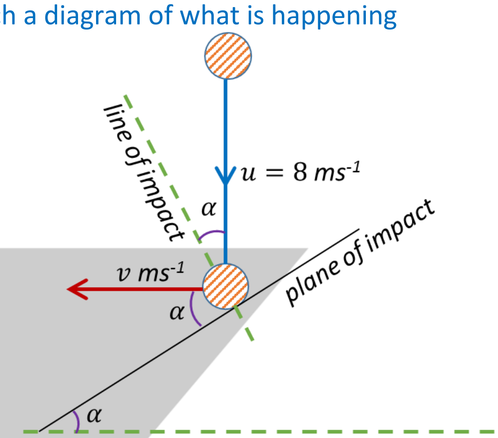
- Impulse affecting each sphere also acts along line of impact
- Components of velocities of spheres along plane of impact unchanged
- Newton's law of restitution applies to components of the velocities of the spheres along line of impact
- The law of conservation of momentum applies along line of impact

{S06-P02} **Question 1:**

A ball drops vertically onto a smooth plane inclined to the horizontal at an angle α. It hits the plane with speed $8ms^{-1}$ and rebounds horizontally. The coefficient of restitution between the ball and the plane is $\frac{1}{3}$. Find the value of α and speed with which the ball rebounds.

Solution:

Sketch a diagram of what is happening



Resolving the initial velocity

Along the plane $8\sin\alpha$ Along the line $8\cos\alpha$

Resolving the final velocity

Along the plane $v\cos\alpha$ Along the line $v\sin\alpha$

The velocity along the plane so can form the equation:

$$8\sin\alpha = v\cos\alpha$$

$$\frac{8}{v}\sin\alpha = \cos\alpha$$

Use velocities along line with coefficient of restitution eqn.

$$e = \frac{1}{3} = \frac{v\sin\alpha}{8\cos\alpha}$$

$$\cos\alpha = \frac{3v}{8}\sin\alpha$$

Equate equations and find v

$$\frac{3v}{8}\sin\alpha = \frac{8}{v}\sin\alpha$$

$$v = \frac{8\sqrt{3}}{3}\ ms^{-1}$$

Use initial equation and find α

$$\frac{\sin\alpha}{\cos\alpha} = \frac{v}{8}$$

$$\alpha = 30°$$

5. Centre of Mass

- **Centre of Mass:** centre of gravity of the system when it is placed in a gravitational field such that each part of system is subject to the same gravitational acceleration
- **Centroid:** geometrical centre; coincides with the centre of mass when the object is made of a uniformly dense material

5.1 Finding Centre of Mass

- We can find the centre of mass by taking moments
- If each mass m_i has position vector $\mathbf{r}_i$ then the position vector of the centre of mass $\bar{\mathbf{r}}$ is

$$\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$$

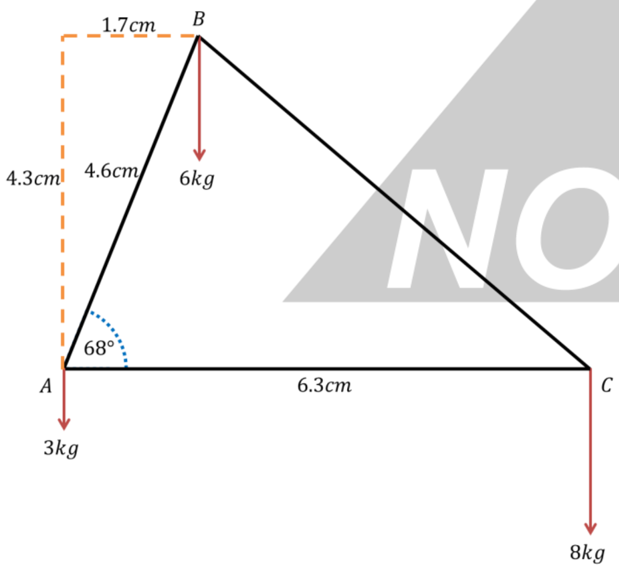
$$\therefore (m_1 + m_2 + \cdots)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = m_1\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots$$

(IM) Ex 11A: **Question 8:**

A light triangular framework ABC has $AB = 4.6$cm, $AC = 6.3$cm and angle $BAC = 68°$. Masses of 3kg, 6kg and 8kg are placed at A, B and C respectively. The framework is suspended from A. Find the angle which AB makes with the vertical.

Solution:

Draw diagram of scenario:



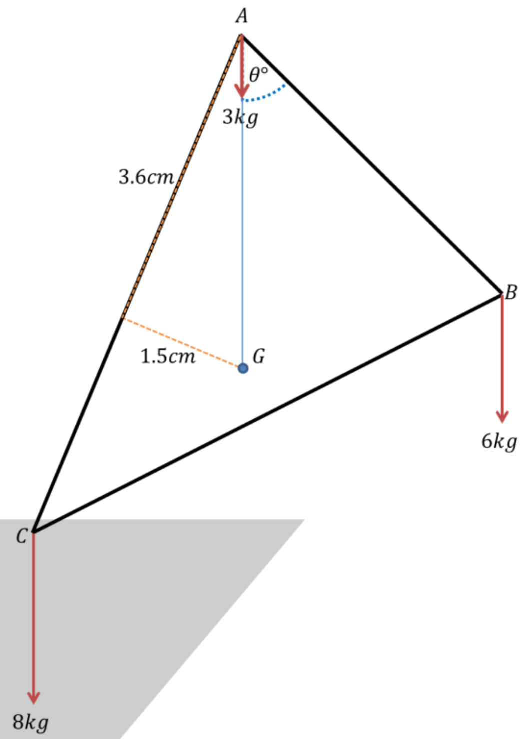
Take A as the origin and find position vectors of B and C using simple Pythagoras theorem:

Substitute into the equation and solve for coordinates:

$$(3+6+8)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3\begin{pmatrix}0\\ 0\end{pmatrix} + 6\begin{pmatrix}1.7\\ 4.3\end{pmatrix} + 8\begin{pmatrix}6.3\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}3.573\\ 1.505\end{pmatrix}$$

Draw diagram as described by scenario:



Find θ using Pythagoras:

$$\theta = 68 - \tan^{-1}\left(\frac{1.5}{3.6}\right)$$

$$\theta = 45.2°$$

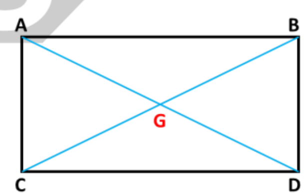
5.2 Centre of Mass of Rigid Bodies

1-Dimensional Objects:

- Uniform rod: centre of mass lies at midpoint of the rod

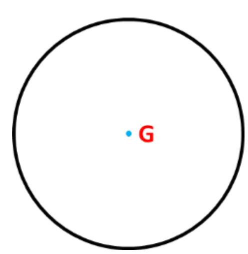
2-Dimensional Objects:

- Uniform Rectangular Lamina: centre of mass is at the intersection of diagonals

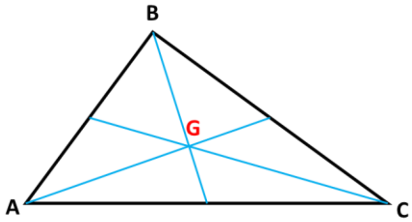


$$G = \begin{pmatrix}midpoint\ of\ AB\\ midpoint\ of\ CD\end{pmatrix}$$

- Uniform Circular Lamina: centre of mass is at the centre of the circle



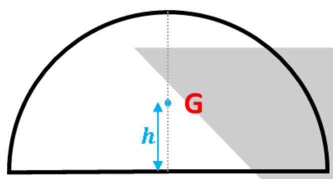
- Uniform Triangular Lamina:



$$A = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, B = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, C = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\therefore G = \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(y_1 + y_2 + y_3) \end{pmatrix}$$

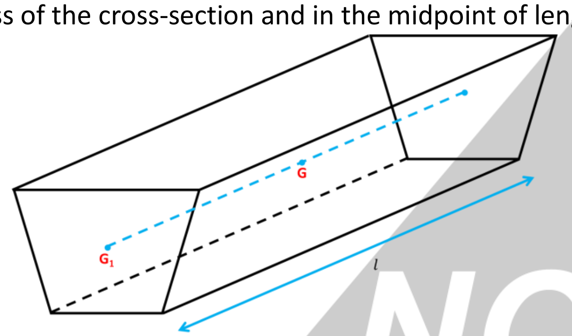
- Uniform Semicircular Lamina:



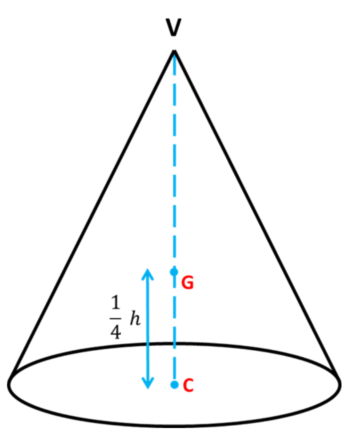
$$h = \frac{4r}{3\pi}$$

3-Dimenionsnal Objects:

- Uniform Solid Prism: centre of mass lies at the centre of mass of the cross-section and in the midpoint of length

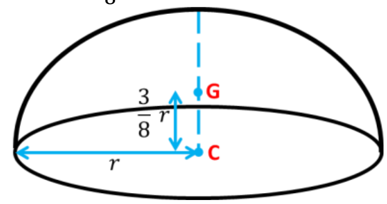


- Uniform Cone: centre of mass lies in the centre of the base and height in ratio 3:1 to its height



$$VG : GC = 3 : 1$$

- Uniform Hemisphere: centre of mass lies in the centre of the base and height $\frac{3}{8}$ of the radius



$$h = \frac{3r}{8}$$

5.3 Composite Bodies

- Split composite body into simple geometrical shapes
- Find the centre of mass of each shape individually
- By taking moments with vectors, find the centre of mass of the composite body
- If the separate geometrical shapes have different densities, use $V \times \rho$ instead of just V
- Therefore in general:

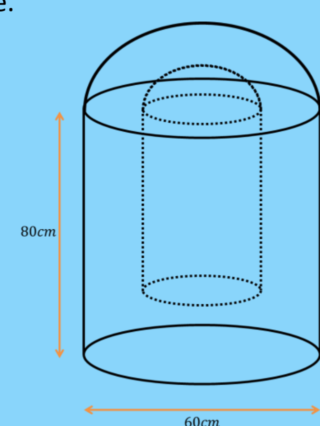
$$(v_1\rho_1 + v_2\rho_2 + \cdots)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = v_1\rho_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v_2\rho_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \cdots$$

- For a lamina with a hole in it, find the centre of mass of the lamina as a whole, then find the centre of mass of the hole and use moments to find the centre of mass of the shaded part
  In these cases:

$$m_{shaded}\begin{pmatrix} x_{shaded} \\ y_{shaded} \end{pmatrix} = m\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - m_{hole}\begin{pmatrix} x_{hole} \\ y_{hole} \end{pmatrix}$$

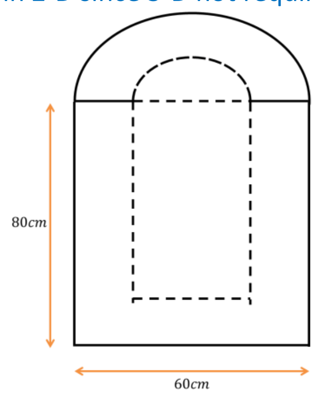
(IM) Ex 11B: **Question 7:**

The diagram shows a box consisting of a cylinder of diameter 60cm and height 80cm, with a hollow cylindrical interior and a hollow hemispherical cap. The thickness of the wall, cap and base is 10cm throughout. Find the height of the centre of mass of the empty box above its base.



Solution:

Draw diagram in 2-D since 3-D not required:



Centre of mass of outside:

For the outer semicircle:

$$h = \frac{3r}{8}$$
$$\frac{3\times 30}{8} = 11.25$$
$$\therefore \textit{height} = 80 + 11.25 = 91.25$$

Then use moments to calculate total outer:

$$72000\pi \begin{pmatrix} 30 \\ 40 \end{pmatrix} + 18000\pi = 90000\pi \begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}$$
$$\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} = \begin{pmatrix} 30 \\ 50.25 \end{pmatrix}$$

Centre of mass of inside:

For the inner semicircle:

$$h = \frac{3r}{8}$$
$$\frac{3\times 20}{8} = 7.5$$
$$\therefore \text{height} = 80 + 7.5 = 87.5$$

Then use moments to calculate the total inner:

$$28000\pi \begin{pmatrix} 30 \\ 45 \end{pmatrix} + \frac{16000}{3}\pi \begin{pmatrix} 30 \\ 87.5 \end{pmatrix} = \frac{100000}{3}\pi \begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix}$$
$$\begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 51.8 \end{pmatrix}$$

Centre of mass of the object:

Use the formula:

$$m_{shaded} \begin{pmatrix} x_{shaded} \\ y_{shaded} \end{pmatrix} = m \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - m_{hole} \begin{pmatrix} x_{hole} \\ y_{hole} \end{pmatrix}$$

Substitute the given scenario into these variables:

$$m_{outer} \begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} = m \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - m_{inner} \begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix}$$
$$90000\pi \begin{pmatrix} 30 \\ 50.25 \end{pmatrix} = \frac{170000}{3}\pi \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - \frac{100000}{3}\pi \begin{pmatrix} 30 \\ 51.8 \end{pmatrix}$$
$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 30 \\ 49.34 \end{pmatrix}$$

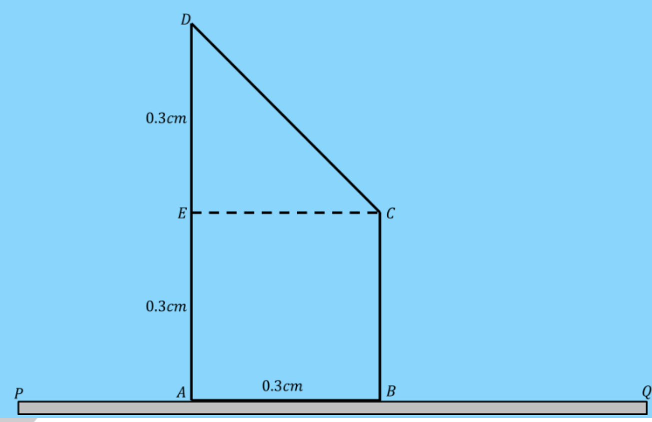
∴ height of centre of mass above its base is 49.34 cm

5.4 Sliding and Toppling

- **Sliding:** when the resultant force on the object parallel to the plane of contact becomes non-zero, that is, the limiting friction force is exceeded by the other forces, the object will slide.
- **Toppling:** when total moment of the forces acting on the object becomes non-zero, the object will topple over.

(IM) Ex 11: **Example 12:**

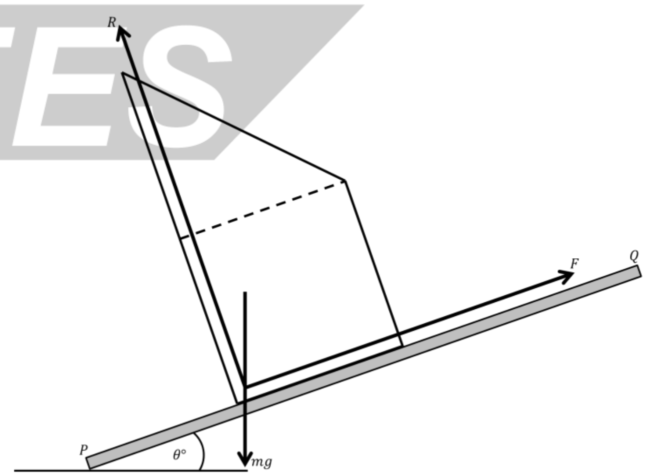
A prism of mass m, having a cross-section as shown, rests in a rough horizontal plank PQ. The coefficient of friction between the plank is 0.4. The end Q of the plank is gradually raised until the equilibrium is broken. Will the prism slide or topple?



Solution:

First find the angle needed to slide:

Draw diagram at hypothetical angle



Resolve forces horizontal to slope:

$$F - mg\sin\theta = 0$$
$$\therefore F = mg\sin\theta$$

Resolve forces vertical to slope:

$$R - mg\cos\theta = 0$$
$$\therefore R = mg\cos\theta$$

Form a relationship by dividing the unknowns:

$$\frac{F}{R} = \tan\theta$$

Substitute $F = 0.4R$ into the equation

$$\tan\theta = 0.4$$
$$\theta = \tan^{-1} 0.4 = 21.8$$

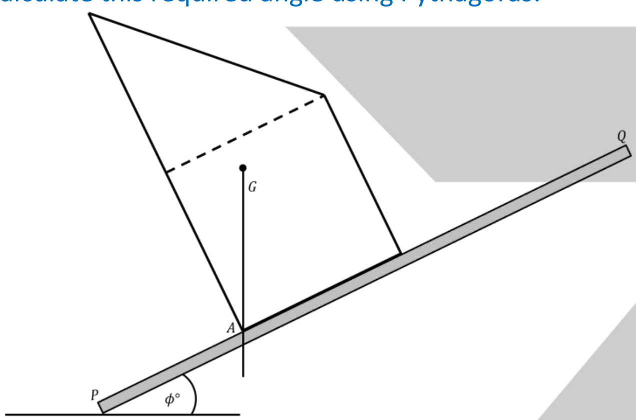
Now find angle needed to topple:

Find centre of mass by using composite bodies rule:

$$m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{2}{3}m\begin{pmatrix}0.15\\ 0.15\end{pmatrix} + \frac{1}{3}m\begin{pmatrix}0.1\\ 0.4\end{pmatrix}$$
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.13\\ 0.23\end{pmatrix}$$

Prism topples when centre of mass is vertically above point A

Calculate this required angle using Pythagoras:



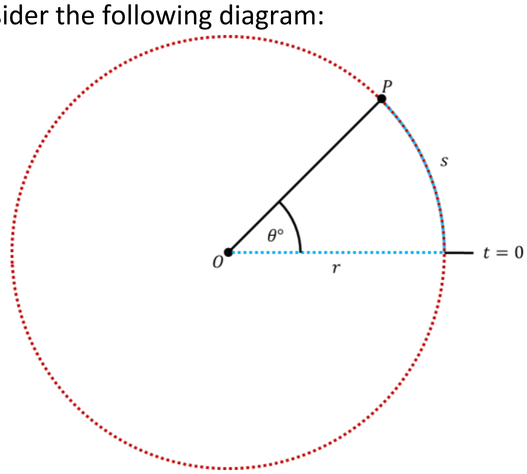
$$\tan\phi = \frac{0.13}{0.23}$$
$$\phi = 29.7$$

From this we can see that $\theta < \phi$, thus when the equilibrium is broken, the object will slide.

6. Circular Motion

6.1 The Basics

- Consider the following diagram:



- Particle P moves in a circular path (the red line)
- **Angular Displacement:**

$$\theta°$$

- **Angular Velocity:**

$$\omega = \frac{2\pi}{T}$$

Where T is the time period for one complete revolution

- **Liner Displacement:**

$$s = \theta r$$

- **Liner Velocity:**

$$v = \omega r$$

And is always tangential to the circle

- **Acceleration:**

$$a = \omega^2 r = \frac{v^2}{r}$$

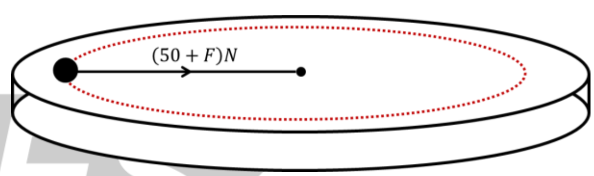
Acts towards the centre of the circle

IM-Ex.16C: **Question 8:**

A particle of mass 3kg is placed on a rough, horizontal turntable and is connected to its centre by a light, inextensible string of length 0.8m. The coefficient of friction between the particle and the turntable is 0.4. The turntable is made to rotate at a uniform speed. If the tension in the string is 50N, find the angular speed of the turntable.

Solution:

Draw diagram of scenario:



Consider the resultant force:

$$D = T + F$$
$$3a = 50 + \mu R$$
$$3a = 50 + 0.4(mg)$$
$$3a = 50 + 0.4(3)(9.81)$$
$$\therefore a = 20.6\ ms^{-2}$$

From this we can find the angular speed of the particle

$$a = \omega^2 r$$
$$\omega^2 r = 20.6$$
$$\omega = \sqrt{\frac{20.6}{0.8}} = 5.07\ rads^{-1}$$

The angular speed of the particle is equal to that of the turntable.

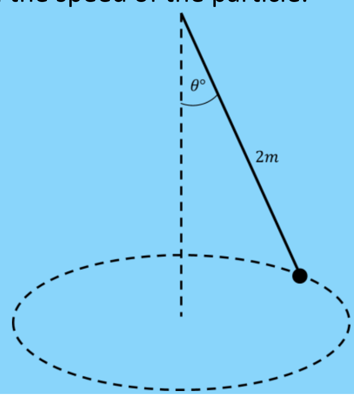
6.2 Conical Pendulum

- In this scenario a body is attached to a fixed point by string and travels in a horizontal circle below that point

{S10-P51}: **Question 3:**

A particle of mass 0.24kg is attached to one end of a light inextensible string of length 2m with the other attached to a fixed point. The particle moves with constant speed in a horizontal circle. The string makes an angle θ with the vertical, and the tension in the string is TN. The acceleration of the particle is 7.5ms^{-2}.

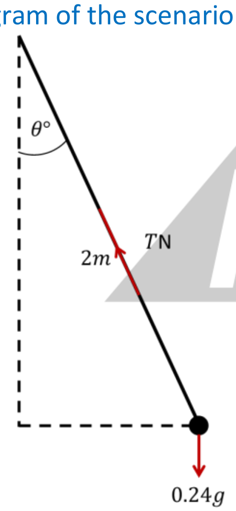
i. Show that $\tan\theta = 0.75$ & find the value of T
ii. Find the speed of the particle.



Solution:

Part (i):

Simplify the diagram of the scenario:



Resolve forces vertically:

$$T\cos\theta = 0.24g$$

Resolve forces horizontally:

$$T\sin\theta = 0.24a$$

Form a relationship by dividing the two equations:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{0.24\times7.5}{0.24\times10}$$

$$\therefore \tan\theta = 0.75$$

Find θ and substitute into original equation to find T:

$$\theta = 36.9°$$

$$T = \frac{0.24\times10}{\cos 36.9} = 3N$$

Part (ii):

Simple algebraic manipulation and Pythagoras:

$$a = \frac{v^2}{r}$$

$$v = \sqrt{ar} = \sqrt{7.5\times2\sin 36.9} = 3\ ms^{-1}$$

6.3 Banked Curves

- Curved sections of public roads/railway tracks banked, enabling cars/trains to travel more quickly around curves
- Normal reaction between the vehicle and the track has a horizontal component when track is banked
- This component helps to provide central force needed to keep vehicle travelling in a circle

(IM) Ex 16D: **Question 10:**

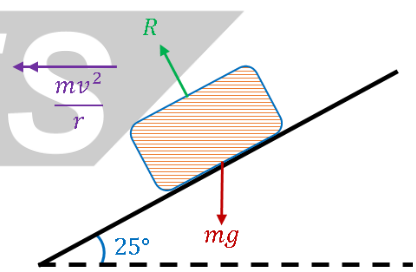
A car is travelling round a curve of radius 150m banked at 25° to the horizontal. The coefficient of friction between the wheels and the road is 0.4.

a) What is the ideal speed of the car round the curve: that is, no lateral frictional force?
b) What is the maximum safe speed of the car?
c) What is the minimum safe speed of the car?

Solution:

Part (a):

Draw diagram of scenario:



Resolve forces vertically:

$$R\cos 25 = mg$$

Resolve forces horizontally:

$$R\sin 25 = \frac{mv^2}{r}$$

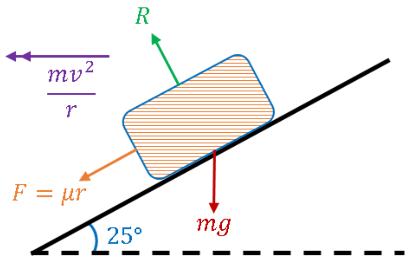
Form a relationship by dividing the two:

$$\tan 25 = \frac{v^2}{150g}$$

$$v = 26.2\ ms^{-1}$$

Part (b):

At max speed, car about to slip upwards ∴ frictional force would act towards the centre of the circle



Resolve forces vertically (direction of gravity):

$$mg + 0.4R\sin 25 = R\cos 25$$

Resolve forces horizontally (direction of centripetal):

$$\frac{mv^2}{r} = 0.4R\cos 25 + R\sin 25$$

Pull out common factors from both sides:

$$m(g) = R(\cos 25 - 0.4\sin 25)$$

$$m\left(\frac{v^2}{150}\right) = R(0.4\cos 25 + \sin 25)$$

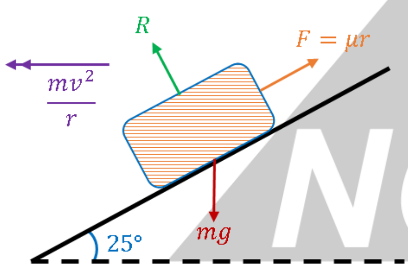
Form a relationship by dividing these equations:

$$\frac{v^2}{150g} = \frac{0.4\cos 25 + \sin 25}{\cos 25 - 0.4\sin 25}$$

$$v = 36.9\ ms^{-1}$$

Part (c):

At min speed, car about to slip downwards ∴ frictional force would act away from the centre of the circle



Resolve forces vertically (direction of gravity):

$$mg = R\cos 25 + 0.4R\sin 25$$

Resolve forces horizontally (direction of centripetal):

$$\frac{mv^2}{r} = R\sin 25 - 0.4R\cos 25$$

Pull out common factors from both sides:

$$m(g) = R(\cos 25 + 0.4\sin 25)$$

$$m\left(\frac{v^2}{150}\right) = R(\sin 25 - 0.4\cos 25)$$

Form a relationship by dividing these equations:

$$\frac{v^2}{150g} = \frac{\sin 25 - 0.4\cos 25}{\cos 25 + 0.4\sin 25}$$

$$v = 9.06\ ms^{-1}$$

6.4 Circular Motion with Non-Uniform Speed

For a particle moving with non-uniform speed:

- The speed of the particle is $|\mathbf{v}| = r\dot{\theta}$ where $\dot{\theta}$ is not constant
- The particle has a **tangential component** of acceleration of $r\ddot{\theta}$
- The particle has a **radial component** of acceleration of $-r\dot{\theta}^2$, which is directed towards the centre of the circle

6.5 Motion in a Vertical Circle

- When moving in a vertical circle a body is in circular motion with non-uniform speed
- **When body cannot leave circle:**
  e.g. body rotating on end of rigid rod
  - If energy of system is sufficient, the body rotates in a complete circle
  - If energy insufficient, the body cannot reach the highest point of the circle and so oscillates between two symmetrical points at each of which its speed is instantaneously zero
- **When body can leave circle:**
  e.g. body rotating on end of string
  - If energy of system is sufficient, the body rotates in a complete circle
  - If energy of system is so low that body cannot rise beyond level of centre of the circle, it oscillates between two symmetrical positions, at each of which its speed is instantaneously zero
  - If energy of system is such that the body can rise above the level of the centre of the circle without being enough to carry it completely around the circle, it will leave the circle and go into projectile motion
- Solve question by using the conservation of energy

7. Elasticity

- Springs can be compressed and stretched
- Elastic strings can only stretch; they go slack

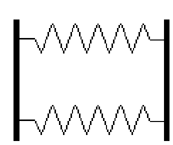
7.1 Hooke's Law

$$T = kx$$

- T is the magnitude of tension
- x is the extension or compression
- k is the spring constant a.k.a. stiffness

- Combining spring constants k
  - Springs in parallel:

$$k_T = k_1 + k_2 + \cdots$$



○ Springs in series:

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

7.2 Modulus of Elasticity

$$T = \frac{\lambda x}{l}$$

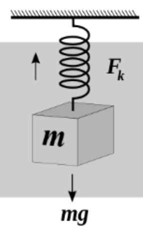
○ λ is the modulus of elasticity
○ l is the natural length

- Relating spring constant k and modulus of elasticity λ:

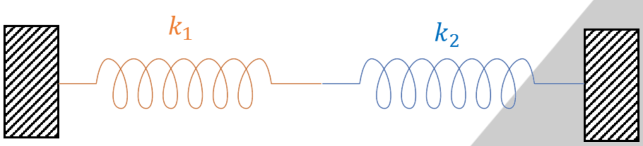
$$\lambda = kl$$

7.3 Scenarios

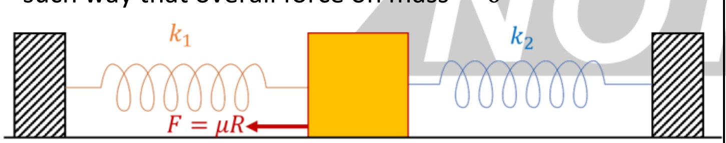
- If a mass is hanging at one end of a spring with the other attached to a fixed point, the tension in the spring must be equal to the weight of the object



- If two springs are attached between two fixed points and both are extended, the tension in both must be the same in order for there to be no net force overall

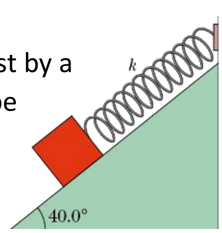


- If two springs are attached to fixed points and an object in the middle, tensions and frictional force must act in such way that overall force on mass $= 0$



- If a spring is attached to an object on a rough surface, the frictional force acts in the direction opposing tension in the spring (preventing it to return to its original shape)

- If a mass is on an incline and held at rest by a spring, the tension in the spring must be equal to the component of the weight parallel to the slope



7.4 Elastic Potential Energy

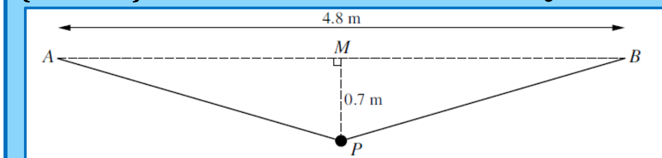
Work done in stretching a string or spring is given by

$$W = \frac{1}{2}kx^2$$

Also gives work done in compressing a spring

- For a scenario, form equation by the conservation of energy and can include e.p.e, k.e, g.p.e and work done against friction

{S10-P51}: **Question 3:**



A particle P of mass 0.35 kg is attached to mid-point of a light elastic string of natural length 4m. Ends of the string are attached to fixed points A & B. P hangs in equilibrium 0.7m vertically below mid-point M of AB.

i. Find tension in the string and hence show that the modulus of elasticity of the string is 25N

P is now held at 1.8m vertically below M, and released

ii. Find speed with which P passes through M

Solution:

Part (i):

Find the extension of the string by finding length AP and PB using right angled triangles

$$AP = PB = \sqrt{2.4^2 + 0.7^2} = 2.5\text{m}$$

$$2.5 + 2.5 - 4 = 1\text{m}$$

The vertical component of tension in the string is equal to the weight as the system is in equilibrium

$$2T\cos\theta = mg$$

$$2T \times \left(\frac{0.7}{2.5}\right) = 0.35 \times 10$$

$$T = 6.25$$

Find the modulus of elasticity by using info calculated

$$T = \frac{\lambda}{l}x$$

$$\lambda = \frac{6.25 \times 4}{1} = 25$$

Part (ii):

Find the spring constant k

$$k = \frac{\lambda}{l} = \frac{25}{4} = 6.25$$

As before, find the extension of the string

$$AP = PB = \sqrt{2.4^2 + 1.8^2} = 3\text{m}$$

$$3 + 3 - 4 = 2\text{m}$$

Form an equation by the conservation of energy, e.p.e = gain in k.e. + gain in g.p.e. + e.p.e at M

$$\left(\frac{1}{2}\times 6.25\times 2^2\right) = \left(\frac{1}{2}\times 0.35\times v^2\right) + (0.35\times 10\times 1.8) + \left(\frac{1}{2}\times 6.25\times 0.8^2\right)$$

$$12.5 = 0.175v^2 + 6.3 + 2$$

$$v^2 = 24$$

$$v = \pm 4.90 \text{ ms}^{-1}$$

8. Oscillatory Motion

8.1 Simple Harmonic Motion

- A particle undergoes simple harmonic motion if its acceleration is directed towards a fixed point and is proportional to the displacement of the particle from that point

$$\ddot{x} = -\omega^2 x$$

8.2 Solutions of the Equation

- Depending upon the scenario the equation of motion can have different solutions
- When particle starts at max. displacement from origin:

$$x = a\cos\omega t$$

- When particle starts at zero displacement from origin:

$$x = a\sin\omega t$$

- When particle starts at intermediate point from origin:

$$x = a\cos(\omega t + \alpha)$$

Note: a outside is different from α inside

8.3 Velocity

- Velocity for simple harmonic motion can be found by differentiating the above solutions

Displacement (x)	Velocity ($\dot{x}$ or v)
$a\cos\omega t$	$-\omega a\sin\omega t$
$a\sin\omega t$	$\omega a\cos\omega t$
$a\cos(\omega t+\alpha)$	$-\omega a\sin(\omega t+\alpha)$

- If given the displacement at a point, the velocity can be found by:

$$v^2 = \omega^2(a^2 - x^2)$$

8.4 Period and Frequency

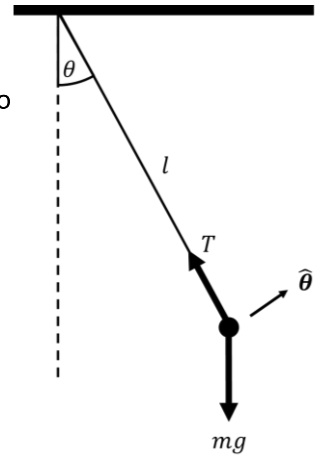
- Oscillation of SHM are periodic and have a time period:

$$T = \frac{2\pi}{\omega}$$

- This can determine the frequency of oscillations:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

8.5 Motion Approximation



- In a simple pendulum, a bob, attached to a rod or string, moves from one side to the other through the equilibrium position
- Assume that:
  - String or rod is inextensible
  - String or rod has no mass
  - Bob is a particle
  - No air resistance
  - No frictional forces
  - Angle of swing is small
- This motion can then be modelled by SHM equations:
  - l is the length of the string or rod
  - m is the mass of the bob
  - θ is the angular displacement
  - $l\ddot{\theta}$ is tangential acceleration
- Resultant force in the tangential direction is:

$$-mg\sin\theta$$

- Using Newton's second law we can derive an equation relating the acceleration and displacement:

$$-mg\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

- When θ is small, $\sin\theta \approx \theta$, causing the equation to be:

$$\ddot{\theta} = -\frac{g}{l}\theta$$

- This is an SHM equation where:

$$\omega^2 = \frac{g}{l}$$

9. Moment of Inertia

9.1 Important Definitions and Formulae

- **Moment of Inertia:** a quantity that expresses a body's tendency to resist angular acceleration; the sum of the products of the mass of each particle in a body with the square of its distance from the axis of rotation

$$I = \sum m_i r_i^2$$

- **Rotational Kinetic Energy:** the kinetic energy of an object due to its rotation; is part of the total kinetic energy of the system

$$E_k = \frac{1}{2}I\dot{\theta}^2$$

where $\dot{\theta}$ is the angular velocity; can be written as ω

- **Moment of Couple:** the rate of change of angular momentum; better known as **torque**

$$C = I\ddot{\theta}$$

where $\ddot{\theta}$ is the angular acceleration
- **Moment of Momentum:** is the rotational analog of linear momentum; better known as **angular momentum**

$$p = I\dot{\theta}$$

9.2 Parallel & Perpendicular Axes Theorems

- **Parallel Axis Theorem:**

$$I_O = I_G + Md^2$$

- **Perpendicular Axis Theorem:**

$$I_z = I_x + I_y$$

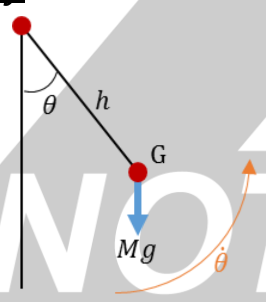
- When solving problems, the axis required may not be the same as that given in the standard results. By combination of symmetry and the theorems above, it is possible to solve and find the moment of inertia
  - E.g. the axis could be perpendicular to a point. For this, find the moment from the centre and then use the parallel theorem to solve
  - E.g. the axis may not be perpendicular. Use the result of the perpendicular axis, symmetry and the perpendicular axis theorem to solve

9.3 Energy in a Rotating Body

- **Total Energy:** sum of rotational kinetic energy of the body and any gravitational potential energy

$$E_T = \frac{1}{2}I\dot{\theta}^2 + Mgh(1 - \cos\theta)$$

where M is the mass of the body, h is the height above original position and θ is the angle that a line connecting the centre of gravity and point of rotation makes with the horizontal



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