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CIE A-LEVEL MATHS 9709 (M2)

FORMULAE AND SOLVED QUESTIONS FOR MECHANICS 1 (M2)

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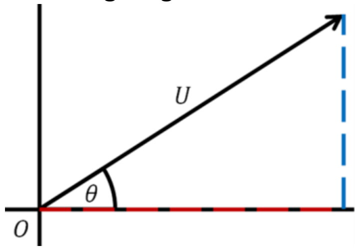
1. Motion of a Projectile

1.1 Initial Velocity

- Consider the following diagram:



- This shows that an initial speed U can be defined in vector notation:

$$\mathbf{u} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j}$$

- In this notation:
  - θ defines the angle of elevation from the horizontal
  - $\mathbf{i}$ defines the horizontal unit vector
  - $\mathbf{j}$ defines the vertical unit vector

1.2 Acceleration

- Gravity affects all projectiles
- Gravity effects only the vertical component of speed
- We can thus define acceleration in vector notation:

$$\mathbf{a} = 0\mathbf{i} - g\mathbf{j} = -g\mathbf{j}$$

- In this notation:
  - g is the acceleration due to gravity
  - $\mathbf{i}$ defines the horizontal unit vector
  - $\mathbf{j}$ defines the vertical unit vector
- Note: this model assumes that air resistance is negligible

1.3 Final Velocity

- This defines the velocity of the projectile after t seconds of travel
- Final velocity, according to the SUVAT equations, is also dependent on the initial velocity and any acceleration acting on the projectile:

$$v = u + at$$

- Above is a scalar equation which can be written as a vector equation:

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

- This can be further simplified into vector notation by substituting previous equations:

$$\mathbf{v} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j} - gt\mathbf{j}$$
$$\mathbf{v} = U\cos\theta\,\mathbf{i} + (U\sin\theta - gt)\mathbf{j}$$

- Substituting any value of t into the above equation produces the velocity of the projectile at that time

1.4 Displacement

- The distance the particle has moved from the origin
- Displacement is dependent on initial velocity and acceleration, or final velocity and acceleration:

$$s = ut + \frac{1}{2}at^2$$
$$s = vt - \frac{1}{2}at^2$$

- These scalar equations can be written as vector equations:

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{r} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

- Substituting previous equations into either of the above equations defines displacement in vector notation:

$$\mathbf{r} = (U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j})t - \frac{1}{2}gt^2\mathbf{j}$$
$$\mathbf{r} = (Ut\cos\theta)\mathbf{i} + \left(Ut\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}$$

- Substituting any value of t into the above equation produces the displacement of the projectile at that time

1.5 Special Events

- **Time at Maximum Vertical Height**: the time at which the projectile reaches its max height, at a given initial velocity and angle of elevation
  - This occurs when vertical component of velocity is 0:

$$U\sin\theta - gt = 0$$
$$\therefore \mathbf{TaMVH} = \frac{U\sin\theta}{g}$$

- **Maximum Vertical Height:** the max height the projectile reaches during its flight
  - This occurs when vertical component of velocity is 0:

$$\mathbf{MVH} = U(\mathbf{TaMVH})\sin\theta - \frac{1}{2}g(\mathbf{TaMVH})^2$$

- **Horizontal Range**: the horizontal distance the projectile covers at a given initial velocity and angle of elevation
  - Occurs when the vertical displacement is 0:

$$Ut\sin\theta - \frac{1}{2}gt^2 = 0$$

  - Find t and substitute into the equation below:

$$\mathbf{HR} = Ut\cos\theta$$

- **Maximum Horizontal Range**: the maximum possible horizontal distance the projectile can cover at a given initial velocity
  - Occurs when the angle of elevation is 45°:

$$\mathbf{MHR} = Ut \times \frac{\sqrt{2}}{2}$$

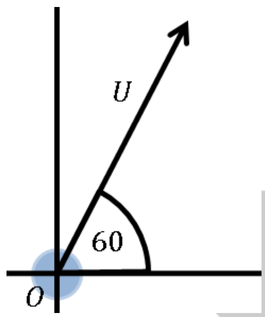
(IM) Ex 7C: **Question 9:**

A ball was projected at an angle of 60° to the horizontal. One second later another ball was projected from the same point at an angle of 30° to the horizontal. One second after the second ball was released, the two balls collided. Show that the velocities of the balls were 12.99ms^{-1} and 15ms^{-1}. Take the value of g to be 10ms^{-2}.

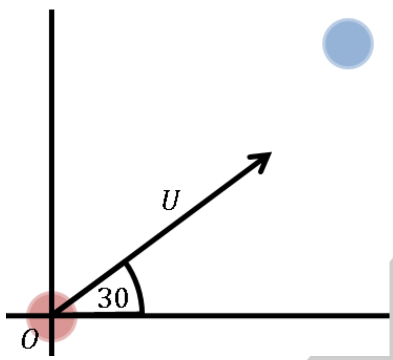
Solution:

Visualise the scenario:

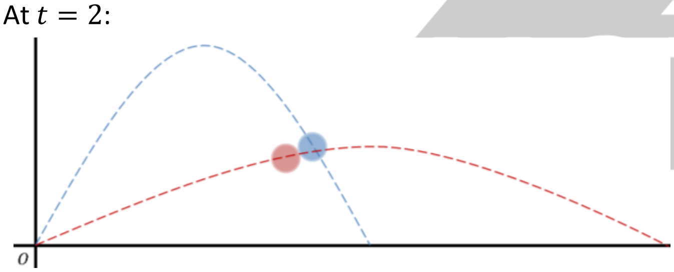
At zero time:



At $t = 1$:



At $t = 2$:



Let velocity for first ball be U_1 and second ball be U_2

Displacement from origin is equal for both at impact

$$\therefore \mathbf{r}_1 = \mathbf{r}_2$$

$$\mathbf{r}_1 = (U_1(2)\cos 60)\mathbf{i} + (U_1(2)\sin 60 - 5(2)^2)\mathbf{j}$$
$$\mathbf{r}_2 = (U_2(1)\cos 30)\mathbf{i} + (U_2(1)\sin 30 - 5(1)^2)\mathbf{j}$$

Equate horizontal components:

$$\therefore 2U_1 \cos 60 = U_2 \cos 30$$

$$U_1 = \frac{\sqrt{3}}{2}U_2$$

Equate vertical components and substitute info above:

$$\therefore 2U_1 \sin 60 - 20 = U_2 \sin 30 - 5$$

$$\left(2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\right)U_2 = 15$$

$$U_2 = 15\text{ms}^{-1}$$

Find U_1

$$U_1 = \frac{\sqrt{3}}{2}(15) = 12.99\text{ms}^{-1}$$

2. Equilibrium of a Rigid Body

2.1 Moment of a Force

$$\text{Moment of a force} = |\mathbf{F}| \times d$$

$|\mathbf{F}|$: magnitude of the force

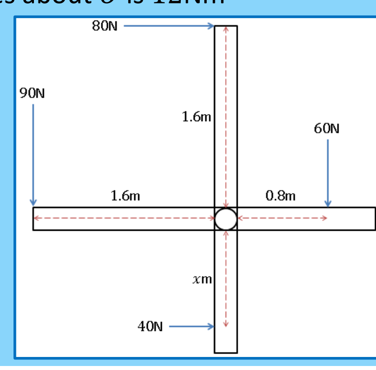
d: perpendicular distance from pivot to point of force

- Units: Newton-meter (Nm)
- Moments are vector quantities and act clockwise or anticlockwise around pivot
- Clockwise is generally considered as positive direction
- **Principle of Moments:** when a system is in equilibrium the sum of anticlockwise moments is equal to the sum of clockwise moments

anti-clockwise moments* = *clockwise moments

(IM) Ex 10A: **Question 4a:**

The diagram shows an aerial view of a revolving door. Four people are exerting forces of 40N, 60N, 80N and 90N as shown. Find the distance x if the total moment of the forces about O is 12Nm



Solution:

Use the sum of turning forces equation:

clockwise moments + anticlockwise moments = 12Nm

Find clockwise moments:

$$(80 \times 1.6) + (60 \times 0.8) = 128 + 48 = 176$$

Find anticlockwise moments:

$$-\big((90 \times 1.6) + (40 \times x)\big) = -(144 + 40x)$$

Substitute back into formula and solve for x:

$$176 + \big(-(144 + 40x)\big) = 12$$
$$176 - 144 - 40x = 12$$
$$x = 0.5\text{m}$$

2.2 Unknown Forces

- Magnitude of forces may not always be given
- Eliminate an unknown/unwanted force by making the point on which it acts the pivot

(IM) Ex 10A: **Question 11:**

A uniform plank is 12m long and has mass 100kg. It is placed on horizontal ground at the edge of a cliff, with 4m of the plank projecting over the edge.

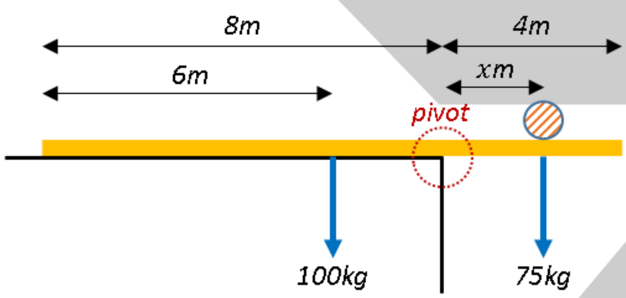
i. How far out from the cliff can a man of mass 75kg safely walk?

ii. The man wishes to walk to the end of the plank. What is the minimum mass he should place on the other end of the plank to do this?

Solution :

Part (i)

Draw up diagram of given scenario:



Use the principle of moments and solve for x:

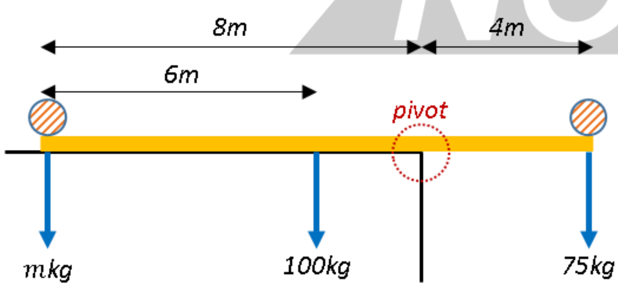
anti-clockwise moments = clockwise moments

$$100g \times 2 = 75g \times x$$

$$\therefore x = \frac{100g \times 2}{75g} = 2.67$$

Part (ii)

Draw up diagram of given scenario:



Minimum mass therefore maximum distance from pivot

Use the principle of moments and solve for x:

anti-clockwise moments = clockwise moments

$$(100g \times 2) + (mg \times 8) = 75g \times 4$$

$$\therefore m = \frac{(75g \times 4) - (100g \times 2)}{g \times 8} = 12.5$$

2.3 Forces in Different Directions

- Forces may act at an angle to the plane
- Equilibrium maintained using components of forces

(IM) Ex 10A: **Question 9:**

A uniform ladder of mass 20kg and length 3m rests against a smooth wall with the bottom of the ladder on smooth horizontal ground and attached by means of a light inextensible string, 1m long, to the base of the wall

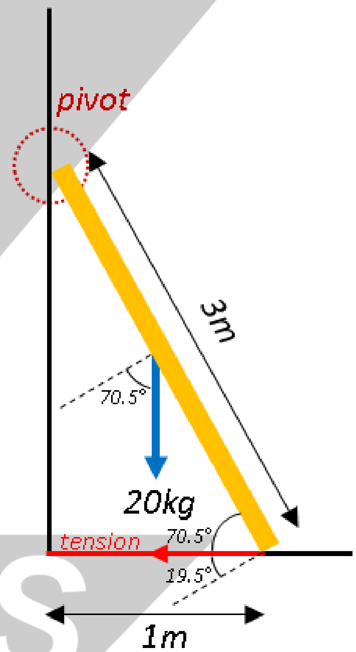
i. Find the tension in the string.

ii. If the breaking strain of the string is 250N, find how far up the ladder a man of mass 80kg can safely ascend.

Solution:

Part (i)

Draw up diagram of given scenario:



Find the angle of elevation, θ, from the horizontal:

$$\cos\theta = \frac{1}{3}$$

$$\theta = 70.5°$$

Use the principle of moments and solve for T:

$$1.5 \times 20g \times \cos 70.5 = 3 \times T \times \cos 19.5$$

$$T = 34.7$$

Part (ii)

In above scenario assume that the man can take any position on the ladder, call it x

Use the principle of moments and solve for x:

anti-clockwise moments = clockwise moments

$$\big((1.5)20g + (x)80g\big)\cos 70.5 = (3)(250)\cos 19.5$$

$$x = 2.32$$

2.4 Centre of Mass

- **Centre of Mass:** centre of gravity of the system when it is placed in a gravitational field such that each part of system is subject to the same gravitational acceleration
- **Centroid:** geometrical centre; coincides with the centre of mass when the object is made of a uniformly dense material

2.5 Finding Centre of Mass

- We can find the centre of mass by taking moments
- If each mass m_i has position vector $\mathbf{r}_i$ then the position vector of the centre of mass $\bar{\mathbf{r}}$ is

$$\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$$

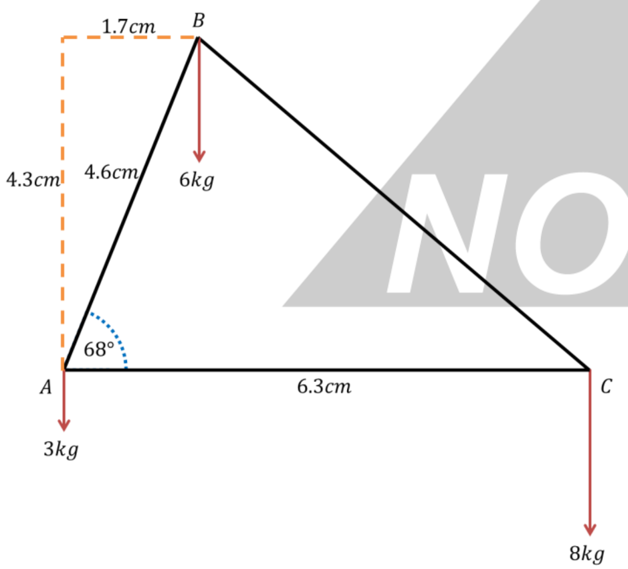
$$\therefore (m_1 + m_2 + \cdots)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = m_1\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots$$

(IM) Ex 11A: **Question 8:**

A light triangular framework ABC has $AB = 4.6$cm, $AC = 6.3$cm and angle $BAC = 68°$. Masses of 3kg, 6kg and 8kg are placed at A, B and C respectively. The framework is suspended from A. Find the angle which AB makes with the vertical.

Solution:

Draw diagram of scenario:



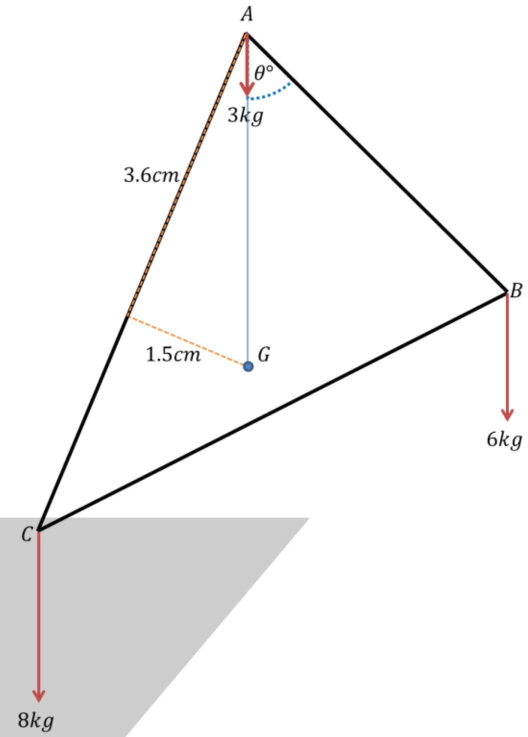
Take A as the origin and find position vectors of B and C using simple Pythagoras theorem:

Substitute into the equation and solve for coordinates:

$$(3 + 6 + 8)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3\begin{pmatrix}0\\ 0\end{pmatrix} + 6\begin{pmatrix}1.7\\ 4.3\end{pmatrix} + 8\begin{pmatrix}6.3\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}3.573\\ 1.505\end{pmatrix}$$

Draw diagram as described by scenario:



Find θ using Pythagoras:

$$\theta = 68 - \tan^{-1}\left(\frac{1.5}{3.6}\right)$$

$$\theta = 45.2°$$

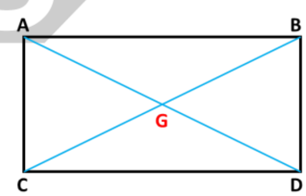
2.6 Centre of Mass of Rigid Bodies

1-Dimensional Objects:

- Uniform rod: centre of mass lies at midpoint of the rod

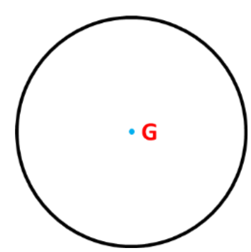
2-Dimensional Objects:

- Uniform Rectangular Lamina: centre of mass is at the intersection of diagonals

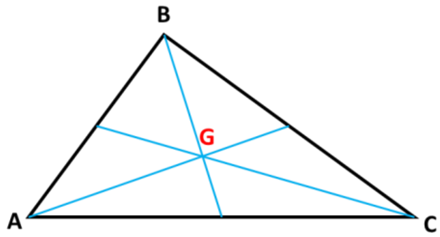


$$G = \begin{pmatrix}midpoint\ of\ AB\\ midpoint\ of\ CD\end{pmatrix}$$

- Uniform Circular Lamina: centre of mass is at the centre of the circle



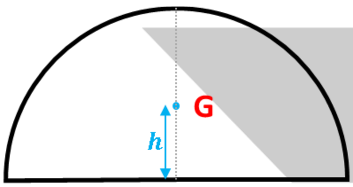
- Uniform Triangular Lamina:



$$A = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, B = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, C = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\therefore G = \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(y_1 + y_2 + y_3) \end{pmatrix}$$

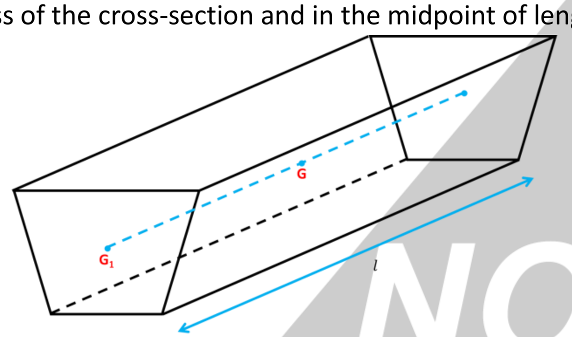
- Uniform Semicircular Lamina:



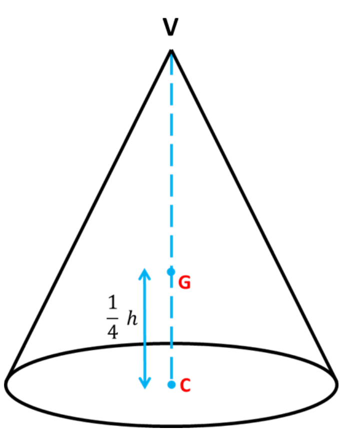
$$h = \frac{4r}{3\pi}$$

3-Dimenionsnal Objects:

- Uniform Solid Prism: centre of mass lies at the centre of mass of the cross-section and in the midpoint of length

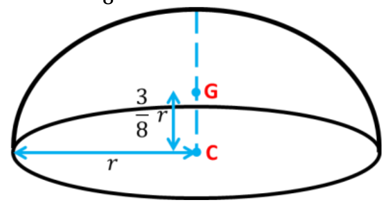


- Uniform Cone: centre of mass lies in the centre of the base and height in ratio 3:1 to its height



$$VG : GC = 3 : 1$$

- Uniform Hemisphere: centre of mass lies in the centre of the base and height $\frac{3}{8}$ of the radius



$$h = \frac{3r}{8}$$

2.7 Composite Bodies

- Split composite body into simple geometrical shapes
- Find the centre of mass of each shape individually
- By taking moments with vectors, find the centre of mass of the composite body
- If the separate geometrical shapes have different densities, use $V \times \rho$ instead of just V
- Therefore, in general:

$$(v_1\rho_1 + v_2\rho_2 + \cdots)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = v_1\rho_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + v_2\rho_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \cdots$$

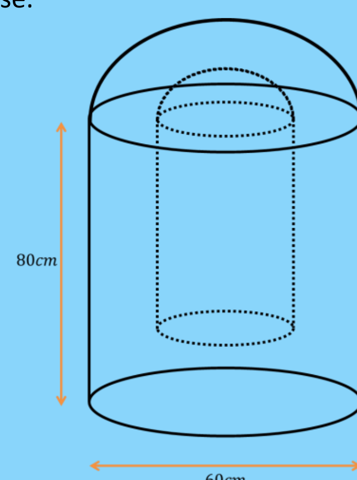
- For a lamina with a hole in it, find the centre of mass of the lamina , then find centre of mass of the hole and use moments to find the centre of mass of the shaded part

In these cases:

$$m_{shaded}\begin{pmatrix} x_{shaded} \\ y_{shaded} \end{pmatrix} = m\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - m_{hole}\begin{pmatrix} x_{hole} \\ y_{hole} \end{pmatrix}$$

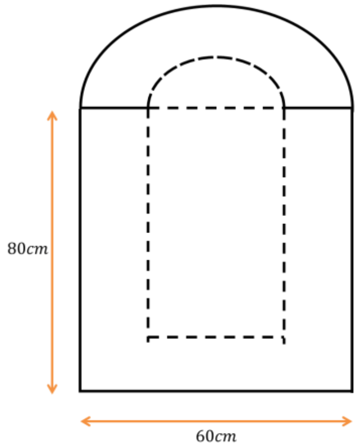
(IM) Ex 11B: **Question 7:**

The diagram shows a box consisting of a cylinder of diameter 60cm and height 80cm, with a hollow cylindrical interior and a hollow hemispherical cap. The thickness of the wall, cap and base is 10cm throughout. Find the height of the centre of mass of the empty box above its base.



Solution:

Draw diagram in 2-D since 3-D not required:



Centre of mass of outside:

For the outer semicircle:

$$h = \frac{3r}{8}$$
$$\frac{3 \times 30}{8} = 11.25$$
$$\therefore height = 80 + 11.25 = 91.25$$

Then use moments to calculate total outer:

$$72000\pi \binom{30}{40} + 18000\pi = 90000\pi \binom{\bar{x}_1}{\bar{y}_1}$$
$$\binom{\bar{x}_1}{\bar{y}_1} = \binom{30}{50.25}$$

Centre of mass of inside:

For the inner semicircle:

$$h = \frac{3r}{8}$$
$$\frac{3 \times 20}{8} = 7.5$$
$$\therefore \text{height} = 80 + 7.5 = 87.5$$

Then use moments to calculate the total inner:

$$28000\pi \binom{30}{45} + \frac{16000}{3}\pi \binom{30}{87.5} = \frac{100000}{3}\pi \binom{\bar{x}_2}{\bar{y}_2}$$
$$\binom{\bar{x}_2}{\bar{y}_2} = \binom{30}{51.8}$$

Centre of mass of the object:

Use the formula:

$$m_{shaded} \binom{x_{shaded}}{y_{shaded}} = m \binom{\bar{x}}{\bar{y}} - m_{hole} \binom{x_{hole}}{y_{hole}}$$

Substitute the given scenario into these variables:

$$m_{outer} \binom{\bar{x}_1}{\bar{y}_1} = m \binom{\bar{x}}{\bar{y}} - m_{inner} \binom{\bar{x}_2}{\bar{y}_2}$$
$$90000\pi \binom{30}{50.25} = \frac{170000}{3}\pi \binom{\bar{x}}{\bar{y}} - \frac{100000}{3}\pi \binom{30}{51.8}$$
$$\binom{\bar{x}}{\bar{y}} = \binom{30}{49.34}$$

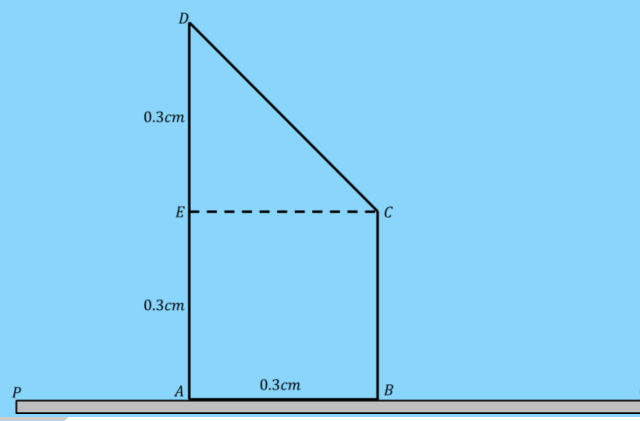
$\therefore$ height of centre of mass above its base is 49.34 cm

2.8 Sliding and Toppling

- **Sliding:** when the resultant force on the object parallel to the plane of contact becomes non-zero, that is, the limiting friction force is exceeded by the other forces, the object will slide.
- **Toppling:** when total moment of the forces acting on the object becomes non-zero, the object will topple over.

(IM) Ex 11: **Example 12:**

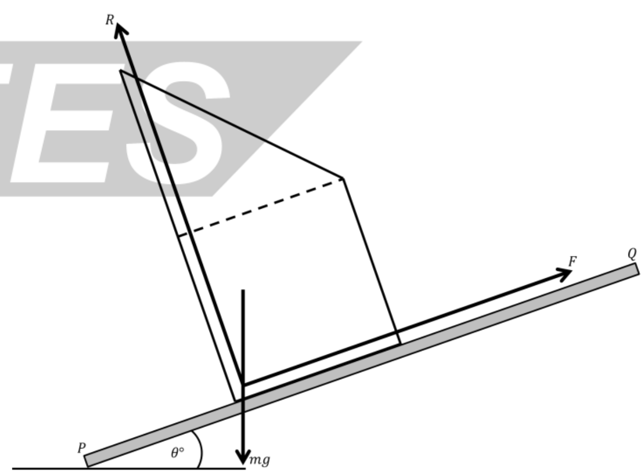
A prism of mass m, having a cross-section as shown, rests in a rough horizontal plank PQ. The coefficient of friction between the plank is 0.4. The end Q of the plank is gradually raised until the equilibrium is broken. Will the prism slide or topple?



Solution:

First find the angle needed to slide:

Draw diagram at hypothetical angle



Resolve forces horizontal to slope:

$$F - mg\sin\theta = 0$$
$$\therefore F = mg\sin\theta$$

Resolve forces vertical to slope:

$$R - mg\cos\theta = 0$$
$$\therefore R = mg\cos\theta$$

Form a relationship by dividing the unknowns:

$$\frac{F}{R} = \tan\theta$$

Substitute $F = 0.4R$ into the equation

$$\tan\theta = 0.4$$
$$\theta = \tan^{-1} 0.4 = 21.8$$

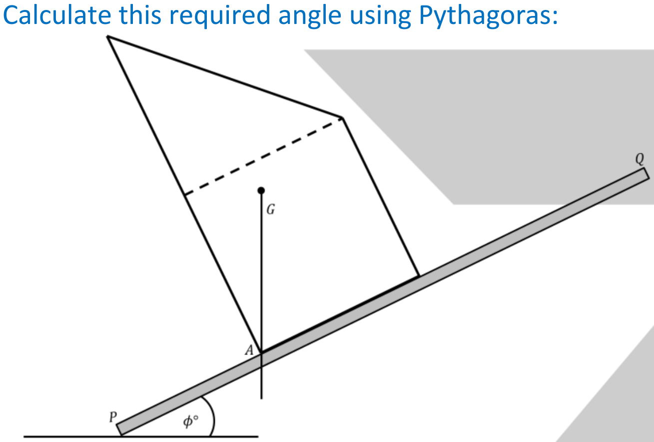
Now find angle needed to topple:

Find centre of mass by using composite bodies rule:

$$m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{2}{3}m\begin{pmatrix}0.15\\ 0.15\end{pmatrix} + \frac{1}{3}m\begin{pmatrix}0.1\\ 0.4\end{pmatrix}$$
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.13\\ 0.23\end{pmatrix}$$

Prism topples when centre of mass is vertically above point A

Calculate this required angle using Pythagoras:



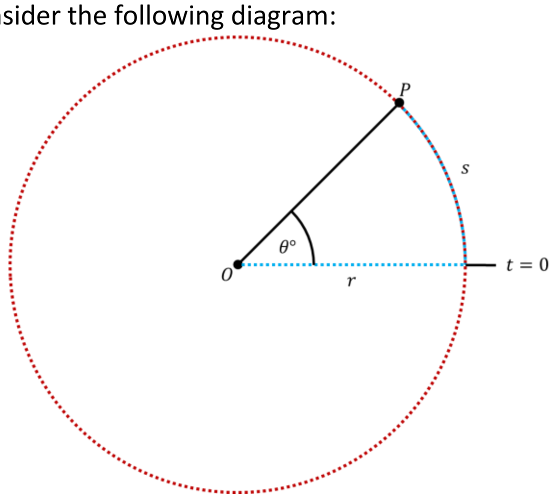
$$\tan\phi = \frac{0.13}{0.23}$$
$$\phi = 29.7$$

From this we can see that $\theta < \phi$, thus when the equilibrium is broken, the object will slide.

3. Uniform Motion in a Circle

3.1 Circular Motion

- Consider the following diagram:



- Particle P moves in a circular path (the red line)
- **Angular Displacement:**

$$\theta°$$

- **Angular Velocity:**

$$\omega = \frac{2\pi}{T}$$

Where T is the time period for one complete revolution

- **Liner Displacement:**

$$s = \theta r$$

- **Liner Velocity:**

$$v = \omega r$$

And is always tangential to the circle

- **Acceleration:**

$$a = \omega^2 r = \frac{v^2}{r}$$

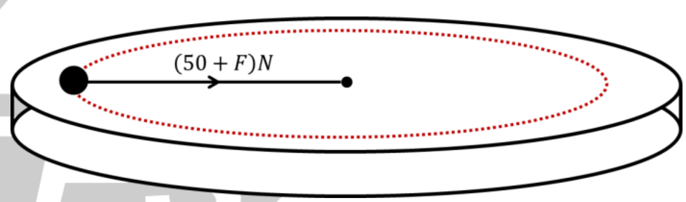
Acts towards the centre of the circle

IM-Ex.16C: **Question 8:**

A particle of mass 3kg is placed on a rough, horizontal turntable and is connected to its centre by a light, inextensible string of length 0.8m. The coefficient of friction between the particle and the turntable is 0.4. The turntable is made to rotate at a uniform speed. If the tension in the string is 50N, find the angular speed of the turntable.

Solution:

Draw diagram of scenario:



Consider the resultant force:

$$D = T + F$$
$$3a = 50 + \mu R$$
$$3a = 50 + 0.4(mg)$$
$$3a = 50 + 0.4(3)(9.81)$$
$$\therefore a = 20.6\ ms^{-2}$$

From this we can find the angular speed of the particle

$$a = \omega^2 r$$
$$\omega^2 r = 20.6$$
$$\omega = \sqrt{\frac{20.6}{0.8}} = 5.07\ rads^{-1}$$

The angular speed of the particle is equal to that of the turntable.

3.2 Horizontal Circles

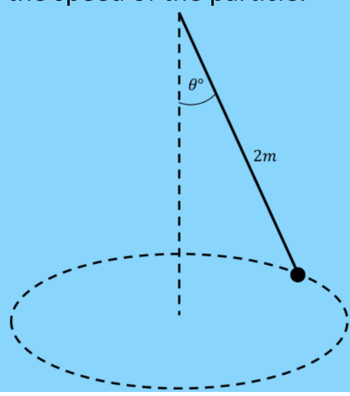
- In this scenario, a body is attached to a fixed point by string and travels in a horizontal circle below that point

{S10-P51}: **Question 3:**

A particle of mass 0.24kg is attached to one end of a light inextensible string of length 2m with the other attached to a fixed point. The particle moves with constant speed in a horizontal circle. The string makes an angle θ with the vertical, and the tension in the string is TN. The acceleration of the particle is 7.5ms^{-2}.

i. Show that $\tan\theta = 0.75$ & find the value of T

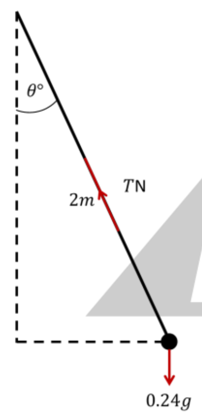
ii. Find the speed of the particle.



Solution:

Part (i):

Simplify the diagram of the scenario:



Resolve forces vertically:

$$T\cos\theta = 0.24g$$

Resolve forces horizontally:

$$T\sin\theta = 0.24a$$

Form a relationship by dividing the two equations:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{0.24\times 7.5}{0.24\times 10}$$

$$\therefore \tan\theta = 0.75$$

Find θ and substitute into original equation to find T:

$$\theta = 36.9°$$

$$T = \frac{0.24\times 10}{\cos 36.9} = 3N$$

Part (ii):

Simple algebraic manipulation and Pythagoras:

$$a = \frac{v^2}{r}$$

$$v = \sqrt{ar} = \sqrt{7.5\times 2\sin 36.9} = 3\,ms^{-1}$$

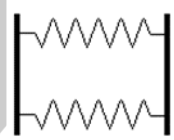
4. Hooke's Law

4.1 Extension & Compression

$$T = kx$$

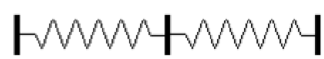
- T is the magnitude of tension
- x is the extension or compression
- k is the spring constant a.k.a. stiffness

- Combining spring constants k
  - Springs in parallel:



$$k_T = k_1 + k_2 + \cdots$$

  - Springs in series:



$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

4.2 Modulus of Elasticity

$$T = \frac{\lambda x}{l}$$

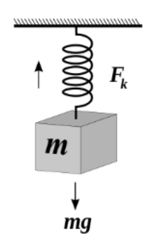
- λ is the modulus of elasticity
- l is the natural length

- Relating spring constant k and modulus of elasticity λ:

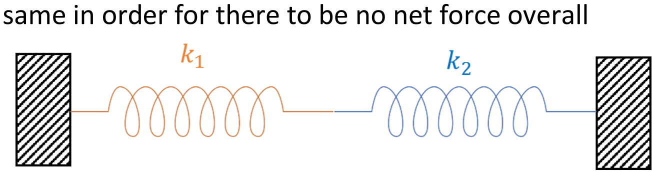
$$\lambda = kl$$

4.3 Scenarios

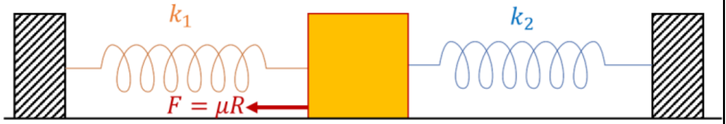
- If a mass is hanging at one end of a spring with the other attached to a fixed point, the tension in the spring must be equal to the weight of the object



- If two springs are attached between two fixed points and both are extended, the tension in both must be the same in order for there to be no net force overall

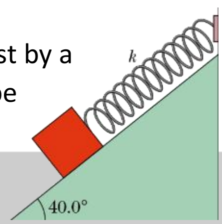


- If two springs are attached to fixed points and an object in the middle, tensions and frictional force must act in such way that overall force on mass $= 0$



- If a spring is attached to an object on a rough surface, the frictional force acts in the direction opposing tension in the spring (preventing it to return to its original shape)

- If a mass is on an incline and held at rest by a spring, the tension in the spring must be equal to the component of the weight parallel to the slope



4.4 Elastic Potential Energy

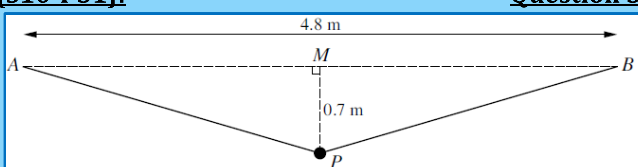
Work done in stretching a string or spring is given by

$$W = \frac{1}{2}kx^2$$

Also gives work done in compressing a spring

- For a scenario, form equation by the conservation of energy and can include e.p.e, k.e, g.p.e and work done against friction

{S10-P51}: **Question 3:**



A particle P of mass 0.35 kg is attached to mid-point of a light elastic string of natural length 4m. Ends of the string are attached to fixed points A & B. P hangs in equilibrium 0.7m vertically below mid-point M of AB.

i. Find tension in the string and hence show that the modulus of elasticity of the string is 25N

P is now held at 1.8m vertically below M, and released

ii. Find speed with which P passes through M

Solution:

Part (i):

Find the extension of the string by finding length AP and PB using right angled triangles

$$AP = PB = \sqrt{2.4^2 + 0.7^2} = 2.5\text{m}$$

$$2.5 + 2.5 - 4 = 1\text{m}$$

The vertical component of tension in the string is equal to the weight as the system is in equilibrium

$$2T\cos\theta = mg$$

$$2T \times \left(\frac{0.7}{2.5}\right) = 0.35 \times 10$$

$$T = 6.25$$

Find the modulus of elasticity by using info calculated

$$T = \frac{\lambda}{l}x$$

$$\lambda = \frac{6.25 \times 4}{1} = 25$$

Part (ii):

Find the spring constant k

$$k = \frac{\lambda}{l} = \frac{25}{4} = 6.25$$

As before, find the extension of the string

$$AP = PB = \sqrt{2.4^2 + 1.8^2} = 3\text{m}$$

$$3 + 3 - 4 = 2\text{m}$$

Form an equation by the conservation of energy,

$$\text{e.p.e} = \text{gain in k.e.} + \text{gain in g.p.e.} + \text{e.p.e at } M$$

$$\left(\frac{1}{2} \times 6.25 \times 2^2\right) = \left(\frac{1}{2} \times 0.35 \times v^2\right) + (0.35 \times 10 \times 1.8) + \left(\frac{1}{2} \times 6.25 \times 0.8^2\right)$$

$$12.5 = 0.175v^2 + 6.3 + 2$$

$$v^2 = 24$$

$$v = \pm 4.90 \text{ ms}^{-1}$$

5. Linear Motion under a Variable Force

5.1 Acceleration as a Derivative

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = v.\frac{dv}{ds}$$

Example:

A particle moves with acceleration, $a = -2v^2$ where v is velocity. Initially, particle at 0 with $v = 2$

i. Find expression for v, in terms of s

ii. Find expression for v, in terms of t

Solution:

Part (i):

Express acceleration as a derivative with displacement

$$v.\frac{dv}{ds} = -2v^2$$

$$v.dv = -2v^2.ds$$

Separate the variables and integrate

$$\int \frac{v}{v^2}.dv = \int -2.ds$$

$$\ln v = -2s + c$$

Substitute given information to find c

$$\ln 2 = 0 + c \therefore c = \ln 2$$

Rearrange found equation to make v the subject:

$$\ln v = -2s + \ln 2$$
$$\ln v - \ln 2 = -2s$$
$$\ln\left(\frac{v}{2}\right) = -2s$$
$$\frac{v}{2} = e^{-2s} \therefore v = 2e^{-2s}$$

Part (ii):

Express acceleration as a derivative with time

$$\frac{dv}{dt} = -2v^2$$
$$v^{-2}.dv = -2.dt$$
$$\int v^{-2}.dv = \int -2.dt$$

Integrate the expression

$$-v^{-1} = -2t + c$$

Substitute given information to find c

$$-(2)^{-1} = -2(0) + c \therefore c = -\frac{1}{2}$$

Rearrange found equation to make v the subject:

$$-v^{-1} = -2t - \frac{1}{2}$$
$$v^{-1} = 2t + \frac{1}{2}$$
$$v = \frac{1}{2t + \frac{1}{2}} = \frac{2}{4t + 1}$$

5.2 Variable Forces

- Bodies undergo variable acceleration due to the effect of variable forces e.g. gravitational fields
- Important to put negative sign if decelerating e.g. when a body falls in resistive medium
- Deriving an expression for force in terms of velocity and displacement:

$$F = ma \qquad a = v\frac{dv}{dx}$$
$$F = mv\frac{dv}{dx}$$

- Deriving an expression for force in terms of work done:

For an object moving from x_1 to x_2 the change in kinetic energy or work done can be defined as:

$$W = \int_{x_1}^{x_2} F\,dx$$
$$\therefore \frac{dW}{dx} = F$$

- Deriving an expression of power in terms of force and velocity

$$P = \frac{dW}{dt} = \frac{dW}{dx} \times \frac{dx}{dt} = Fv$$

(IM) Ex: 9Misc **Question 10:**

A car of mass 1200kg is travelling on a straight horizontal road, with its engine working at a constant rate of 25kW. Given that the resistance to motion of the car is proportional to the square of its velocity and that the greatest constant speed the car can maintain is 50ms^{-1}, show that $125000 - v^3 = 6000v^2\frac{dv}{dx}$, where v ms^{-1} is the velocity of the car when its displacement from a fixed point on the road is x metres.

Solution:

Write down known facts:

$$P = 25000 \qquad F = kv^2 \qquad m = 1200$$

Find k:

$$F = kv^2$$
$$\frac{P}{v} = kv^2$$
$$\frac{25000}{50} = k(50^2)$$
$$k = 0.2$$

Write an equation that relates the forces:

$$Engine\ force = Work\ force + Resistive\ force$$
$$\frac{P}{v} = mv\frac{dv}{dx} + kv^2$$
$$\frac{25000}{v} = 1200v\frac{dv}{dx} + \frac{1}{5}v^2$$
$$\frac{125000}{v} = 6000v\frac{dv}{dx} + v^2$$
$$125000 = 6000v^2\frac{dv}{dx} + v^3$$

Hence find distance covered by the car in increasing its speed from 30 ms^{-1} to 45 ms^{-1} by forming an integral to define displacement in terms of velocity:

$$\frac{dv}{dx} = \frac{125000 - v^3}{6000v^2}$$
$$\frac{dx}{dv} = \frac{6000v^2}{125000 - v^3}$$
$$dx = \frac{6000v^2}{125000 - v^3}dv$$
$$x = \int \frac{6000v^2}{125000 - v^3}dv$$

Substitute the info we know:

$$x = \int_{30}^{45} \frac{6000v^2}{125000 - v^3}dv$$
$$x = 6000\int_{30}^{45} \frac{v^2}{125000 - v^3}dv$$
$$x = 6000\left[-\frac{\ln(125000 - v^3)}{3}\right]_{30}^{45} = 2125$$

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