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CIE A-LEVEL

MATHS 9709 (P1)

FORMULAE AND SOLVED QUESTIONS FOR PURE 1 (P1)

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1. QUADRATICS

1.1 Completing the square

$$x^2 + nx \iff \left(x + \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$
$$a(x+n)^2 + k$$

Where the vertex is $(-n, k)$

1.2 Sketching the Graph

- y-intercept
- x-intercept
- Vertex (turning point)

1.3 Discriminant

$$b^2 - 4ac$$

If $b^2 - 4ac = 0$, real and equal roots
If $b^2 - 4ac < 0$, no real roots
If $b^2 - 4ac > 0$, real and distinct roots

1.4 Quadratic Inequalities

$$(x-d)(x-\beta) < 0 \Longrightarrow d < x < \beta$$
$$(x-d)(x-\beta) > 0 \Longrightarrow x < d \text{ or } x > \beta$$

1.5 Solving Equations in Quadratic Form

- To solve an equation in some form of quadratic
- Substitute y
- E.g. $2x^4 + 3x^2 + 7$, $y = x^2$, $\therefore 2y^2 + 3y + 7$

2. FUNCTIONS

Domain = x values & Range = y values

- One-one functions: one x-value gives one y-value

2.1 Find Range

- Complete the square or differentiate
- Find min/max point
- If min then, $y \geq \min y$
- If max then, $y \leq \max y$

2.2 Composition of 2 Functions

- E.g. $fg(x) \Longrightarrow f(g(x))$

2.3 Prove One-One Functions

- One x value substitutes to give one y value
- No indices

2.4 Finding Inverse

- Make x the subject of formula

2.5 Relationship of Function & its Inverse

- The graph of the inverse of a function is the reflection of a graph of the function in $y = x$

{W12-P11} **Question 10:**

$$f(x) = 4x^2 - 24x + 11, \text{ for } x \in \mathbb{R}$$
$$g(x) = 4x^2 - 24x + 11, \text{ for } x \leq 1$$

i. Express $f(x)$ in the form $a(x-b)^2 + c$, hence state coordinates of the vertex of the graph $y = f(x)$
ii. State the range of g
iii. Find an expression for $g^{-1}(x)$ and state its domain

Solution:

Part (i)

First pull out constant, 4, from x related terms:

$$4(x^2 - 6x) + 11$$

Use following formula to simplify the bracket only:

$$\left(x - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$
$$4[(x-3)^2 - 3^2] + 11$$
$$4(x-3)^2 - 25$$

Part (ii)

Observe given domain, $x \leq 1$.
Substitute highest value of x

$$g(x) = 4(1-3)^2 - 25 = -9$$

Substitute next 3 whole numbers in domain:

$$x = 0, -1, -2 \qquad g(x) = 11, 23, 75$$

Thus they are increasing

$$\therefore g(x) \geq -9$$

Part (iii)

Let $y = g(x)$, make x the subject

$$y = 4(x-3)^2 - 25$$
$$\frac{y+25}{4} = (x-3)^2$$
$$x = 3 + \sqrt{\frac{y+25}{4}}$$

Can be simplified more

$$x = 3 \pm \frac{1}{2}\sqrt{y+25}$$

Positive variant is not possible because $x \leq 1$ and using positive variant would give values above 3

$$\therefore x = 3 - \frac{1}{2}\sqrt{y+25}$$
$$\therefore g^{-1}(x) = 3 - \frac{1}{2}\sqrt{x+25}$$

Domain of $g^{-1}(x)$ = Range of $g(x)$ $\therefore x \geq -9$

3. Coordinate Geometry

3.1 Length of a Line Segment

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3.2 Gradient of a Line Segment

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

3.3 Midpoint of a Line Segment

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

3.4 Equation of a Straight Line

- $y = mx + c$
- $y - y_1 = m(x - x_1)$

3.5 Special Gradients

- Parallel lines: $m_1 = m_2$
- Perpendicular lines: $m_1 m_2 = -1$
- The gradient at any point on a curve is the gradient of the tangent to the curve at that point
- The gradient of a the tangent at the vertex of a curve is equal to zero – stationary point

{Wxx-Pxx} **Question 10:**

Point R is a reflection of point $(-1,3)$ in the line

$$3y + 2x = 33.$$

Find by calculation the coordinates of R

Solution:

Find equation of line perpendicular to $3y + 2x = 33$ intersecting point $(-1,3)$

$$3y + 2x = 33 \;\Leftrightarrow\; y = 11 - \frac{2}{3}x$$

$$m = -\frac{2}{3}$$

$$m.m_1 = -1 \text{ and so } m_1 = \frac{3}{2}$$

Perpendicular general equation:

$$y = \frac{3}{2}x + c$$

Substitute known values

$$3 = \frac{3}{2}(-1) + c \text{ and so } c = \frac{9}{2}$$

Final perpendicular equation:

$$2y = 3x + 9$$

Find point of intersection by equating two equations

$$11 - \frac{2}{3}x = \frac{3x + 9}{2}$$

$$13 = \frac{13}{3}x$$

$$x = 3, \qquad y = 9$$

Vector change from $(-1,3)$ to $(3,9)$ is the vector change from $(3,9)$ to R

Finding the vector change:

$$\textit{Change in } x = 3 - -1 = 4$$

$$\textit{Change in } y = 9 - 3 = 6$$

Thus R

$$R's\;\; x = 3 + 4 = 7 \text{ and } R's\;\; y = 9 + 6 = 15$$

$$R = (7,15)$$

4. Circular Measure

4.1 Radians

$$\pi = 180° \text{ and } 2\pi = 360°$$

$$\text{Degrees to radians: } \times \frac{\pi}{180}$$

$$\text{Radians to degrees: } \times \frac{180}{\pi}$$

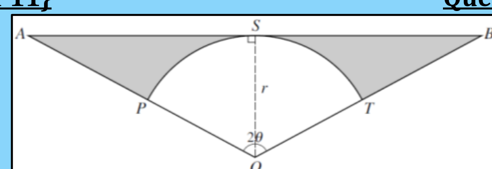
4.2 Arc length

$$s = r\theta$$

4.3 Area of a Sector

$$A = \frac{1}{2}r^2\theta$$

{S11-P11} **Question 9:**



Triangle OAB is isosceles, $OA = OB$ and ASB is a tangent to PST

i. Find the total area of shaded region in terms of r and θ

ii. When $\theta = \frac{1}{3}$ and $r = 6$, find total perimeter of shaded region in terms of $\sqrt{3}$ and π

Solution:

Part (i)

Use trigonometric ratios to form the following:

$$AS = r\tan\theta$$

Find the area of triangle OAS:

$$OAS = \frac{r\tan\theta \times r}{2} = \frac{1}{2}r^2\tan\theta$$

Use formula of sector to find area of OPS:

$$OPS = \frac{1}{2}r^2\theta$$

Area of ASP is $OAS - OPS$:

$$\therefore ASP = \frac{1}{2}r^2\tan\theta - \frac{1}{2}r^2\theta = \frac{1}{2}r^2(\tan\theta - \theta)$$

Multiply final by 2 because BST is the same and shaded is ASP and BST

$$Area = 2\times\frac{1}{2}r^2(\tan\theta - \theta) = r^2(\tan\theta - \theta)$$

Part (ii)

Use trigonometric ratios to get the following:

$$\cos\left(\frac{\pi}{3}\right) = \frac{6}{AO}$$
$$\therefore AO = 12$$

Finding AP:

$$AP = AO - r = 12 - 6 = 6$$

Finding AS:

$$AS = 6\tan\left(\frac{\pi}{3}\right) = 6\sqrt{3}$$

Finding arc PS:

$$Arc\ PS = r\theta$$
$$PS = 6\times\frac{\pi}{3} = 2\pi$$

Perimeter of 1 side of the shaded region:

$$Pe_1 = 6 + 6\sqrt{3} + 2\pi$$

Perimeter of entire shaded region is just double:

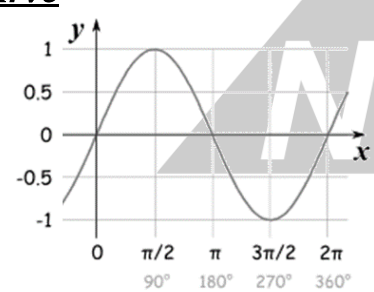
$$2\times Pe_1 = 12 + 12\sqrt{3} + 4\pi$$

5. Trigonometry

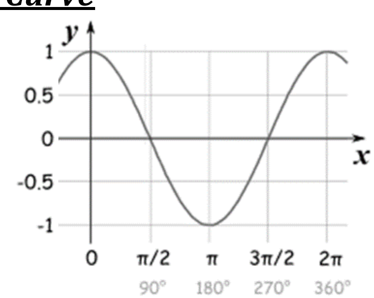
$$y = a(\ldots bx + c) - d$$

- a: changes amplitude
- b: increases no. of cycles
- c: alters x-axis by $-c$
- d: alters y-axis by d

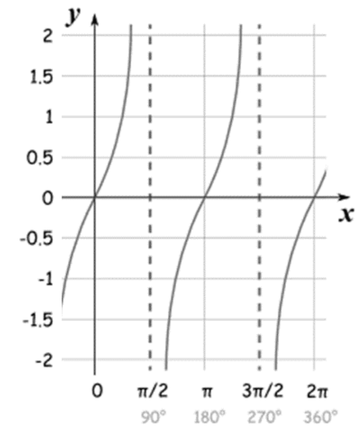
5.1 Sine Curve



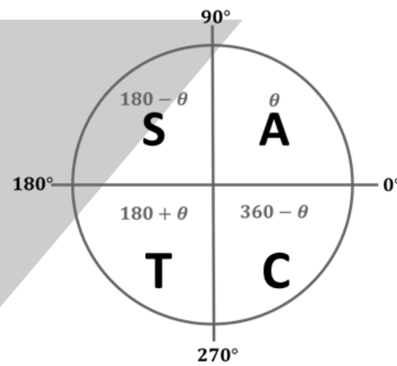
5.2 Cosine Curve



5.3 Tangent Curve



5.4 When sin, cos and tan are positive



5.5 Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta \equiv 1$$

6. Vectors

- Forms of vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \overrightarrow{AB} \qquad \boldsymbol{a}$$

- Position vector: position relative to origin $\overrightarrow{OP}$
- Magnitude $= \sqrt{x^2 + y^2}$
- Unit vectors: vectors of magnitude $1 = \frac{1}{|AB|}\overrightarrow{AB}$
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
- Dot product: $(a\mathbf{i} + b\mathbf{j}).(c\mathbf{i} + d\mathbf{j}) = (ac\mathbf{i} + bd\mathbf{j})$
- $\cos A = \frac{a.b}{|a||b|}$

{S03-P01} **Question 8:**

Points A, B, C, D have position vectors $3\boldsymbol{i} + 2\boldsymbol{k}, 2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}, 2\boldsymbol{j} + 7\boldsymbol{k}, -2\boldsymbol{i} + 10\boldsymbol{j} + 7\boldsymbol{k}$ respectively

i. Use a scalar product to show that BA and BC are perpendicular
ii. Show that BC and AD are parallel and find the ratio of length of BC to length of AD

Solution:

Part (i)

First find the vectors representing BA and BC:

$$BA = OA - OB$$
$$BA = 3\boldsymbol{i} + 2\boldsymbol{k} - (2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k})$$
$$BA = -1\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$
$$CB = OB - OC$$
$$CB = 2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} - (2\boldsymbol{j} + 7\boldsymbol{k})$$
$$CB = 2\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

Now use the dot product rule:

$$BA.CB = 0$$
$$\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$
$$= (-1\times2) + (-2\times-4) + (3\times-2) = 0$$

Thus proving they are perpendicular since $\cos 90 = 0$

Part (ii)

Find the vectors representing BC and AD:

$$BC = -CB$$
$$BC = -\begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$
$$BC = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$
$$AD = OD - OA$$
$$AD = -2\boldsymbol{i} + 10\boldsymbol{j} + 7\boldsymbol{k} - (3\boldsymbol{i} + 2\boldsymbol{k})$$
$$AD = -5\boldsymbol{i} + 10\boldsymbol{j} + 5\boldsymbol{k} = \begin{pmatrix} -5 \\ 10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Direction vector shows that they are parallel

Calculate lengths of each:

$$|BC| = 2\left(\sqrt{(-1)^2 + 2^2 + 1^2}\right) = 2\sqrt{6}$$
$$|AD| = 5\left(\sqrt{(-1)^2 + 2^2 + 1^2}\right) = 5\sqrt{6}$$
$$\therefore |AD|:|BC| = 5:2$$

7. SERIES

7.1 Binomial Theorem

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$$

$${}^nC_r = \frac{n(n-1)(n-2)\dots(n-(r-1))}{r!}$$

7.2 Arithmetic Progression

$$u_k = a + (k-1)d$$

$$S_n = \frac{1}{2}n[2a + (n-1)d]$$

7.3 Geometric Progression

$$u_k = ar^{k-1}$$

$$S_n = \frac{a(1-r^n)}{(1-r)} \qquad S_\infty = \frac{a}{1-r}$$

{W05-P01} **Question 6:**

A small trading company made a profit of $250 000 in the year 2000. The company considered two different plans, plan A and plan B, for increasing its profits. Under plan A, the annual profit would increase each year by 5% of its value in the preceding year. Under plan B, the annual profit would increase each year by a constant amount $\$D$

i. Find for plan A, the profit for the year 2008
ii. Find for plan A, the total profit for the 10 years 2000 to 2009 inclusive
iii. Find for plan B the value of D for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for plan A

Solution:

Part (i)

Increases is exponential ∴ it is a geometric sequence: 2008 is the 9th term:

$$\therefore u_9 = 250000\times1.05^{9-1} = 369000 \text{ (3s.f.)}$$

Part (ii)

Use sum of geometric sequence formula:

$$S_{10} = \frac{250000(1-1.05^{10})}{1-1.05} = 3140000$$

Part (iii)

Plan B arithmetic; equate 3140000 with sum formula

$$3140000 = \frac{1}{2}(10)(2(250000) + (10-1)D)$$
$$D = 14300$$

8. DIFFERENTIATION

When $y = x^n, \frac{dy}{dx} = nx^{n-1}$

- 1st Derivative = $\frac{dy}{dx} = f'(x)$
- 2nd Derivative = $\frac{d^2y}{dx^2} = f''(x)$
- Increasing function: $\frac{dy}{dx} > 0$
- Decreasing function: $\frac{dy}{dx} < 0$
- Stationary point: $\frac{dy}{dx} = 0$

8.1 Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

8.2 Nature of Stationary Point

- Find second derivative
- Substitute x-value of stationary point
- If value +ve → min. point
- If value –ve → max. point

8.3 Connected Rates of Change

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

{W05-P01} **Question 6:**

The equation of a curve is given by the formula:

$$y = \frac{6}{5 - 2x}$$

i. Calculate the gradient of the curve at the point where $x = 1$

ii. A point with coordinates (x, y) moves along a curve in such a way that the rate of increase of y has a constant value of 0.02 units per second. Find the rate of increase of x when $x = 1$

Solution:

Part (i)

Differentiate given equation

$$6(5 - 2x)^{-1}$$

$$\frac{dy}{dx} = 6(5 - 2x)^{-2} \times -2 \times -1$$

$$= 12(5 - 2x)^{-2}$$

Now we substitute the given x value:

$$\frac{dy}{dx} = 12\big(5 - 2(1)\big)^{-2}$$

$$\frac{dy}{dx} = \frac{4}{3}$$

Thus the gradient is equal to $\frac{4}{3}$ at this point

Part (ii)

Rate of increase in time can be written as:

$$\frac{dx}{dt}$$

We know the following:

$$\frac{dy}{dx} = \frac{4}{3} \quad and \quad \frac{dy}{dt} = 0.02$$

Thus we can formulate an equation:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Rearranging the formula we get:

$$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx}$$

Substitute values into the formula

$$\frac{dx}{dt} = 0.02 \div \frac{4}{3}$$

$$\frac{dx}{dt} = 0.02 \times \frac{3}{4} = 0.015$$

9. INTEGRATION

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

Definite integrals: substitute coordinates and find 'c'

9.1 To Find Area

- Integrate curve
- Substitute boundaries of x
- Subtract one from another (ignore c)

$$\int_c^d y \, dx$$

9.2 To Find Volume

- Square the function
- Integrate and substitute
- Multiply by π

$$\int_c^d \pi y^2 \, dx$$

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