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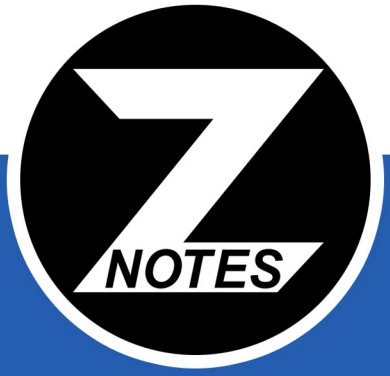


COLLEGEBOARD SAT II MATHS (LEVEL 2)

GUIDE & SUMMARIZED NOTES ON THE SYLLABUS

TABLE OF CONTENTS

NUMBER & OPERATIONS

1. OPERATIONS

1.1 Unknown Operation

- Questions introduce new symbol to denote mathematical operations
- Think of the symbol as a $+$, $-$, $\times$, or $\div$
- Similar idea to a function but not exactly

Example:

- $x \blacksquare y = 4x + \frac{y^2}{2}$
- Given that $n \blacksquare 4 = 12$
- Find the value of n
- $n \blacksquare 4 = 4n + \frac{4^2}{2} = 12$
- $\therefore 4n + 8 = 12$
- $4n = 4$
- $n = 1$

1.2 Rearrangement

- Solving for y in terms of x can also be asked
  - Rearrange the given equation(s) to make y the subject
  - x must be the only other unknown

2. RATIO & PROPORTION

- **Proportion:** equation that sets 2 fractions equal to each other
- Cross multiplication is used to find unknowns

3. COMPLEX NUMBERS

3.1 Form

- Written in the form $a + bi$
  - a is the real part of the complex number
  - b is the imaginary part of the complex number
  - i is the constant: $\sqrt{-1}$
- Complex numbers are often denoted as a letter
- Note: $i^2 = -1$

3.2 Arithmetic

- **Addition or subtraction:** real with real, imaginary with imaginary
- **Multiplication or division:** proceed with i as an unknown, as if x was in same position

3.3 Complex Conjugates

- $a + bi$ is the complex conjugate of $a - bi$ and vice versa
- $(a + bi)(a - bi) = a^2 + b^2$

3.4 Calculating the Magnitude

Example:

- $z = 3 - 4i$
- $\therefore |z| = |3 - 4i|$
- Here the 2 lines denote the magnitude of the complex number
- Simply: $|z| = \sqrt{3^2 + 4^2} = 5$

4. COUNTING

4.1 Fundamental Counting Principle

- Action 1 can be done in x number of ways
- Action 2 can be done in y number of ways
- The number of ways the 2 actions can be done in order is $x \times y$

4.2 Permutations & Combinations

- **Permutations:** an arrangement of elements where order matters
- **Combinations:** an arrangement of elements where order does not matter

5. ELEMENTARY NUMBER THEORY

5.1 Types of Numbers

- Positive numbers
- Negative numbers
- Prime numbers
- Integers
- Odd and Even numbers

5.2 Example

- If x is an odd number and y an even number
- Given several options find the operation that results in an odd number
- Substitute suitable numbers meeting the criteria above to find the correct option

5.3 Logical Questions

- Options are conditional statements: if $1 + 1$ is 2, then $1 + 2$ is 3
- Question would ask to select the correct or incorrect statement
- Could also be true or false scenario

6. Matrices

- **Matrix:** a rectangular array of numbers written in brackets
  - If a matrix has a dimension denoted by $a \times b$
  - It has a rows and b columns
- **Element:** a number within the matrix
- 2 matrices are equal when dimensions AND corresponding elements are equal
- Addition and subtraction: add and subtract corresponding elements
- **Identity Matrix:** often denoted by I
  - $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

6.1 Multiplication of Matrices

- **Scalar Product:** multiply a constant outside the matrix with all the elements of it
  - E.g.: $a\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & aw \end{pmatrix}$
- **Product of 2 matrices:** multiply rows of first matrix by corresponding columns of second
  - E.g.: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (ax + by) \\ (cx + dy) \end{pmatrix}$
- NOTE: in latter sometimes multiplication not possible due to incompatible dimensions

6.2 Determinant

- **Determinant:** an integer associated with each matrix
  - E.g.: $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  - $\therefore \det X = ad - bc$

7. Sequence and Series

- A set of numbers in a certain order
- Can be finite or infinite:
  - **Finite:** sequence ends after a certain number of numbers
  - **Infinite:** sequence continues endlessly

7.1 Arithmetic Sequence

- Consecutive numbers share a common ***DIFFERENCE***
- **Formula for *n*th term:**

$$u_n = a + (n - 1)d$$

- **Formula for sum of a finite arithmetic sequence:**

$$S_n = \frac{1}{2}n[a + l]$$

$$S_n = \frac{1}{2}n[2a + (n - 1)d]$$

7.2 Geometric Sequence

- Consecutive terms share a common ***RATIO***
- **Formula for *n*th term:**

$$u_n = ar^{n-1}$$

- **Formula for sum of geometric sequences:**

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

- **Formula for sum to infinity of geometric sequences:**

$$S_\infty = \frac{a}{1 - r}$$

8. Vectors

- A physical quantity that has both magnitude and direction (think of Despicable Me)
- **Resultant Vector:** basic arithmetic of 2 vectors
  - E.g.: $v = (v_1, v_2)$ and $u = (u_1, u_2)$
  - $\therefore R = (v_1 + u_1, v_2 + u_2)$

8.1 Norm (Magnitude)

Norm: the magnitude or length of the vector

- If $R = (v_1 + u_1, v_2 + u_2)$
- Then $\|R\| = \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2}$
- Note: norm can be represented as a single or double bar
- If norm equals 1 then the vector is a unit vector

8.2 Dot Product

- E.g.: $v.u = v_1u_1 + v_2u_2$
- If dot product equals 0 then the vectors are perpendicular to each other
- Result is a real number not another vector

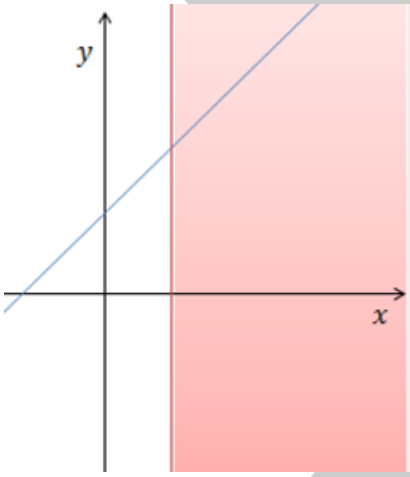
ALGEBRA & FUNCTIONS

1. EXPRESSIONS, EQUATIONS AND INEQUALITIES

- **Expression:** algebraic arithmetic that represents a value
  - E.g.: $x + 2$
  - Cannot be drawn on the Cartesian Plane
- **Equation:** algebraic arithmetic that defines an unknown
  - E.g.: $y = x + 2$
  - Can be drawn on the Cartesian Plane
- **Inequality:** an algebraic arithmetic that limits what be
  - E.g.: $x > 1.5$
  - Can be drawn on the Cartesian Plane

1.1 Equation and Inequality on a Graph

- Consider the following graph:



  - The blue line represents $y = x + 2$
  - The red shaded area represents $x > 1.5$
  - NOTE: x cannot lie on the red line

1.2 Functions

- The same as an equation but with limits
- **Domain:** x values of a functions
- **Range:** y values of a function
- **Composite**: $f(g(x))$; find value of $g(x)$ then substitute into f
- **Inverse:** $f^{-1}(x)$; make x subject of formula, change x to $f^{-1}(x)$ and $f(x)$ into x

1.3 Completing the Square

$$x^2 + nx \Leftrightarrow \left(x + \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

$$a(x + n)^2 + k$$

Where the vertex is $(-n, k)$

1.4 Discriminant

$$b^2 - 4ac$$

If $b^2 - 4ac = 0$, real and equal roots

If $b^2 - 4ac < 0$, no real roots

If $b^2 - 4ac > 0$, real and distinct roots

2. POLYNOMIALS

- These have several unknowns to several degrees of powers

2.1 Identities

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a + b)(a - b)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

2.2 Binomial Theorem

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_n y^n$$

2.3 Factor and Remainder Theorem

- **Factor theorem:** If $(x - t)$ is a factor of the function $p(x)$ then $p(t) = 0$
- **Remainder theorem:** If the function $f(x)$ is divided by $(x - t)$ then the remainder: $R = f(t)$

3. LOGARITHMIC AND EXPONENTIAL FUNCTIONS

$$y = a^x \Leftrightarrow \log_a y = x$$

$$\log_a 1 = 0 \qquad \log_a a = 1$$
$$\log_a b^n \equiv n \log_a b$$
$$\log_a b + \log_a c \equiv \log_a bc$$
$$\log_a b - \log_a c \equiv \log_a \frac{b}{c}$$
$$\log_a b \equiv \frac{\log b}{\log a}$$
$$\log_a b \equiv \frac{1}{\log_b a}$$

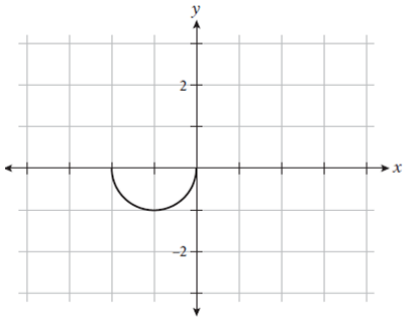
4. OTHER FUNCTIONS

4.1 Periodic

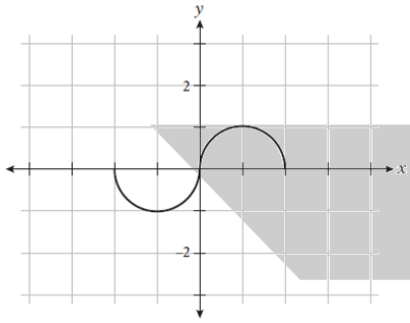
- Every x in domain there is some constant c such that:

$$f(x + c) = f(x)$$

- E.g.: A function f is defined to give the following graph:



- If it is described to have a period of 2 for the interval $-2 < x < 2$, then the graph would be:

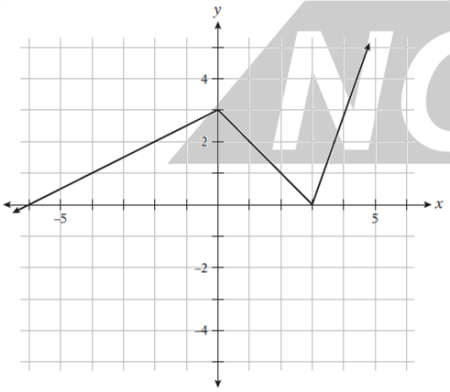


4.2 Piecewise

- It has different rules for different intervals of its domain
- E.g.:

$$f(x) = \begin{cases} x < 0, & \frac{1}{2x} + 3 \\ 0 \le x \le 3, & -x + 3 \\ x > 3, & 3x - 9 \end{cases}$$

- The graph would be:



4.3 Recursive

- Defines terms of a sequence by relating each to previous terms
- E.g.: the Fibonacci sequence

$$a_n = a_{n-2} + a_{n-1}$$

- Conditions: $n \ge 2,\ a_0 = 1,\ a_1 = 1$

4.4 Parametric

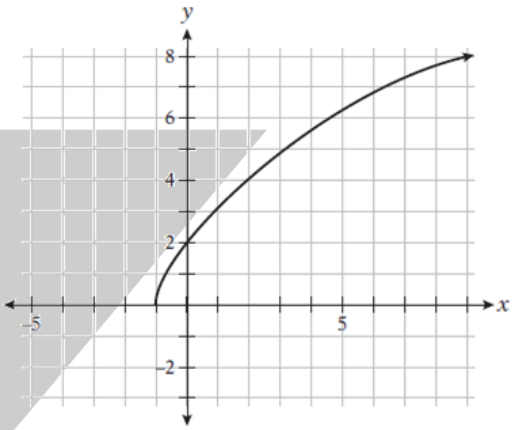
- Where x and y are defined in terms of another third variable
- This variable is called the parameter
- E.g.:

$$x = t^2 - 1 \qquad y = 3t$$

- A table showing values of t, x, y:

t	0	1	2	3
x	-1	0	3	8
y	0	3	6	9

- The graph of the equation:



GEOMETRY & MEASUREMENT

1. SOLID GEOMETRY

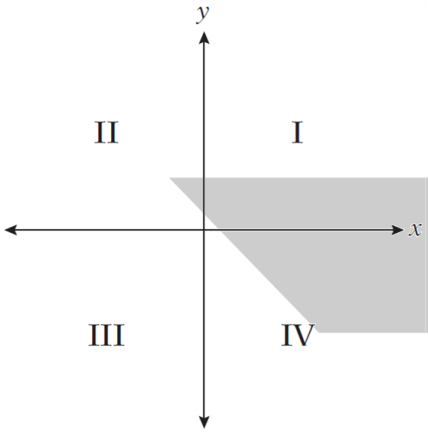
1.1 Relevant Vocabulary

- **Faces**: flat surfaces of a polyhedron shaped like polygons
- **Edge**: a segment where two faces intersect
- **Vertex**: point of intersection of three or more edges
- **Base:**
  - Prism or Cylinder: the two congruent, parallel faces
  - Pyramid or Cone: the polygonal face that does not contain the common vertex
- **Lateral faces**: face(s) that make up sides of the solid
  - Prisms: lateral faces are always parallelograms
- **Altitude**: the perpendicular segment…
  - Prism or Cylinder: …to plane of both bases
  - Pyramid or Cone: …joining the vertex to the plane of the base
- **Height** (h): the length of the altitude
- **Slant height** (l): the distance from the edge of the base to the common vertex

- **Apothem:** length from the center of a regular polygon at right-angles to any of its sides
- **Volume:** amount of space enclosed by a solid
- **Surface Area:** sum of areas of all faces of a solid
- **Lateral Area:** often called the lateral surface area, sum of areas of only lateral (vertical) faces

2. Coordinate Geometry

2.1 Cartesian Plane



- Roman numerals represent each sector, although not often shown on the graph itself
- **Abscissa:** the x-coordinate
- **Ordinate:** the y-coordinate

2.2 Length of a Line Segment

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2.3 Gradient of a Line Segment

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- **Parallel Lines:** have slopes that are equal, $m_1 = m_2$
- **Perpendicular Lines:** have slopes that are opposite reciprocals, $m_1 m_2 = -1$

2.4 Midpoint of a Line Segment

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2.5 Equations of Straight Lines

Slope-Intercept Form: $y = mx + c$

Point-Slope Form: $y - y_1 = m(x - x_1)$

Standard Form: $Ax + By = C$

3. Curved Lines

3.1 Circle

$$(x - h)^2 + (y - k)^2 = r^2$$

- (h, k) is the center of the circle and r is its radius

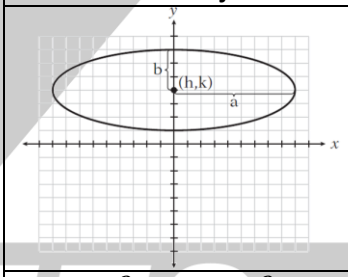
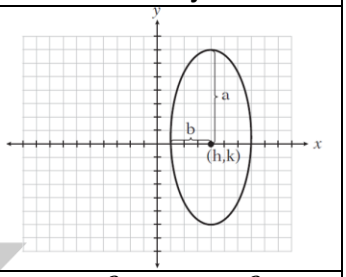
3.2 Parabolas

$$y - k = a(x - h)^2$$

- (h, k) is the turning point of the curve; max or min
- Axis of symmetry exists at $x = h$
- **Shape of curve:**
  - $a > 0$ the graph opens upwards
  - $a < 0$ the graph opens downwards
- Change positions of x and y with each other, parabolas will open sideways

3.3 Ellipses

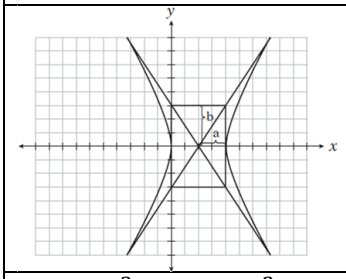
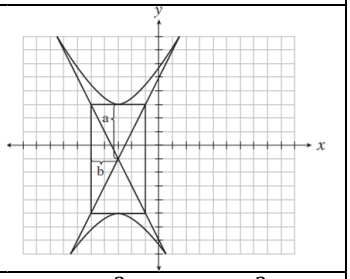
- Set of all points whose sum of distances from 2 fixed points are constant
- *Foci:* the 2 fixed points

Horizontal Major Axis	Vertical Major Axis
	
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

- (h, k) is the center of the ellipse
- **Lengths of the Ellipses:**
  - $2a$ is the largest length of the ellipse
  - $2b$ is the smallest length of the ellipse
- c is distance from each focus point where, $c^2 = a^2 - b^2$
- **Eccentricity:** $e = \frac{c}{a} < 1$
  - $e \to 1$ ellipse becomes more elongated
  - e gets smaller, ellipse becomes more circular

3.4 Hyperbolas

- Set of all points whose difference of distances from 2 fixed points are constant

Horizontal Transverse Axis	Vertical Transverse Axis
	
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-h)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1$

- (h, k) is the center of the hyperbola
- **Lengths of the Ellipses:**
  - $2a$ is the length of line connecting 2 curved vertices
  - $2b$ is the length of the conjugate
- c is distance from each focus point where, $c^2 = a^2 + b^2$
- **Eccentricity:** $e = \frac{c}{a} > 1$
  - $e \to 1$ hyperbola branches of hyperbola become more pointed and closer together
  - e gets larger, hyperbola becomes nearly flat

4. Symmetry

4.1 Types of Symmetry

- Symmetry with the y-axis
- Symmetry with the x-axis
- Symmetry with the origin

4.2 Tests for Symmetry

- Replace x with $-x$ in equation returns to original equation
- Replace y with $-y$ in equation returns to original equation
- Replace x and y with $-x$ and $-y$ in equation returns to original equation

5. Transformations

5.1 Reflection

- $f(-x)$: reflection in the y-axis
- $-f(x)$: reflection in the x-axis

5.2 Translation

- $f(x) + a$: translation of a units parallel to y-axis
- $f(x + a)$: translation of $-a$ units parallel to x-axis

5.3 Stretch

- $f(ax)$: stretch, scale factor $\frac{1}{a}$ parallel to x-axis
- $af(x)$: stretch, scale factor a parallel to y-axis

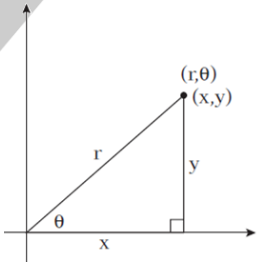
6. Polar Coordinates

- **Rectangular Coordinate System:** (x, y)
- **Polar Coordinate System:** (r, θ)

6.1 Correlation of Coordinate and Polar

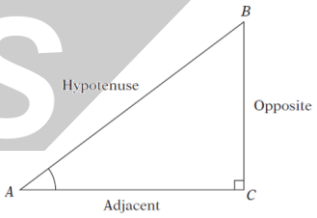
- $x = r\cos\theta$
- $y = r\sin\theta$
- $x^2 + y^2 = r^2$

6.2 Graphical Representation



7. Trigonometry

7.1 The Ratios



Main Functions:

- **Sine:**

$$\sin(BAC) = \frac{opposite}{hypotenuse}$$

- **Cosine:**

$$\cos(BAC) = \frac{adjacent}{hypotenuse}$$

- **Tangent:**

$$\tan(BAC) = \frac{opposite}{adjacent}$$

Reciprocal Functions:

- **Secant:**

$$\sec(BAC) = \frac{1}{\cos(BAC)}$$

- **Cosecant:**

$$\csc(BAC) = \frac{1}{\sin(BAC)}$$

- **Cotangent:**

$$\cot(BAC) = \frac{1}{\tan(BAC)}$$

Inverse Functions:

- **Arcsine:**

$$\sin^{-1}\left(\frac{opposite}{hypotenuse}\right) = BAC$$

- **Arccosine:**

$$\cos^{-1}\left(\frac{adjacent}{hypotenuse}\right) = BAC$$

- **Arctangent:**

$$\tan^{-1}\left(\frac{opposite}{adjacent}\right) = BAC$$

Co-function Identities:

- **Sine and Cosine:**

$$\sin(BAC) = \cos(90 - BAC)$$

- **Tangent and Cotangent:**

$$\tan(BAC) = \cot(90 - BAC)$$

- **Secant and Cosecant:**

$$\sec(BAC) = \csc(90 - BAC)$$

7.2 Trigonometric Identities

- **Basic Identities:**

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

- **Law of Cosines:**

$$c^2 = a^2 + b^2 - 2ab\cos C$$

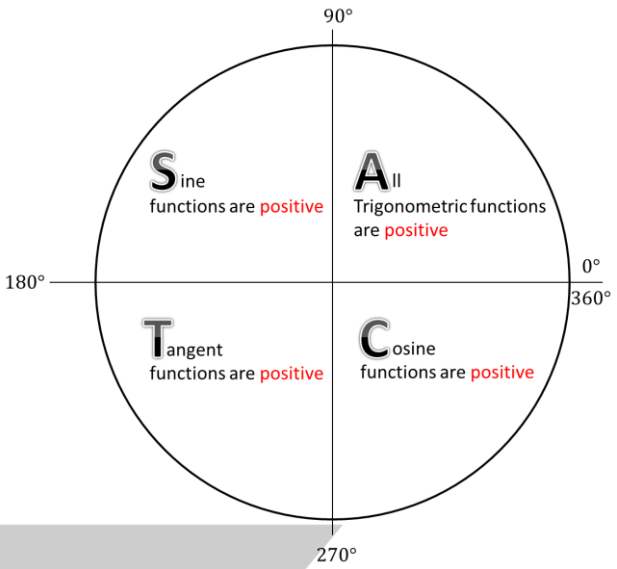
- **Law of Sines:**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- **Double Angle Formulas:**

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

7.3 Trigonometric Circle



- **Radians to angles:**
  - π radians (rads) $= 180°$

DATA ANALYSIS, STATISTICS & PROBABILITY

1. MEASURE OF LOCATION

1.1 Mode

- Most common or most popular data value
- Only average that can be used for qualitative data
- Not suitable if the data values are very varied
- **Modal class:** class with highest frequency density

1.2 Median

- Middle value when data ordered
  - If n odd, median $= {}^1/_2\,(n+1)^{th}$ value
  - If n even, median $= {}^1/_2\,n^{th}$ value
- Not affected be extreme values

1.3 Mean

- Sum of data divided by number of values

$$\bar{x} = \frac{\sum x_i}{n} \quad \text{or} \quad \bar{x} = \frac{\sum x_i f_i}{\sum f_i}$$

- Important as it uses all the data values
- Disadvantage: affected by extreme values
- If data is grouped – use mid-point of group as x

1.4 Range and Quartiles

- **Range:** difference in value of largest and smallest numbers in a set of numbers
- **Upper Quartile (75%):** median value in upper half of a set of ordered data
- **Lower Quartile (25%):** median value in lower half of a set of ordered data
- **Interquartile Range:** difference in values of upper and lower quartiles

2. STANDARD DEVIATION

- Average deviation of all values in a set from the mean

$$\sigma = \sqrt{\frac{\text{Sum of Squares of Deviations from Mean}}{\text{Number of Terms in the Data Set}}}$$

$$\sigma = \sqrt{\frac{1}{n}\sum(x_i - \bar{x})^2} \quad \text{or} \quad \sqrt{\frac{1}{n}\sum x_i^2 - \bar{x}^2}$$

3. REGRESSION

3.1 Linear

- Equation of data follows general rule:

$$y = mx + c$$

- Graph: data points' line of best fit = straight line

3.2 Quadratic

- Equation of data follows a general rule:

$$y = ax^2 + bx + c$$

- Graph: data points have a curve of best fit = parabola
  - ∪-shaped: a is positive
  - ∩-shaped: a is negative

3.3 Exponential

- Equation of data follows a general rule: $y = ab^x$
- Graph: data points will have a curve of best fit

4. PROBABILITY

- It is a ratio between 1 and 0:
  - 1 represents an outcome that is inevitable
  - 0 represents an outcome that is impossible
- The ratio represents the probability of an outcome from a selection of outcomes
- Thus most logical outcomes have ratios in between this

$$\text{Probability} = \frac{\text{Successful Outcomes}}{\text{Total Number of Possible Outcomes}}$$

COLLEGEBOARD SAT II MATHS (L2)